Calculus

from Limits to Integration

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Limits

1. The Behavior of Limits

1.1 Why Are We Doing This?

It is the bane of every Calculus teacher's existence - a dreaded and often-despised phrase that wakes us in a cold sweat in the middle of the night:

"When are we ever going to use this?"

If we can indulge the author for a moment, he can attempt to explain the answer to that question. To not bury the lede here, the entirety of the third chapter of this book is dedicated to uses for Calculus, and more of them are sprinkled throughout the rest of the text. Calculus is used in literally any STEM field on a daily basis, with the last two of those really seeing derivatives and integrals show up everywhere. Calculus is the study of change and accumulation, and there is nothing more constant in life than change.

Can a computer do math better than we can? Most of the time, yes. Who programs the computers, though? Is it ever something that is going to personally come up in your usual outside-of-work day-to-day life? Of course not, but if that argument held we'd stop teaching math at fifth grade, and humans would've never advanced past the pre-industry era. Remember, just because something isn't personally important doesn't mean it isn't important to our whole culture, and we're all part of the culture!

Moreover, the author would make the argument that Calculus, and all of Mathematics as a whole, is the very best way to mold a developing brain to process logic and build critical thinking skills. So while taking a derivative doesn't help us drive a car, the way doing it wires our brain may well help us make better decisions while we're out on the road.

Finally, and this is perhaps the tragedy of the current structure of the whole subject, but the actual payoff for studying Mathematics doesn't come until the end of an undergraduate degree, or perhaps even later! The subject itself is endlessly beautiful and should be done for its own merit and enjoyment, but the way in which it is taught beginning at age five through adulthood is absolutely as mind-numbing and creatively void as it could possibly be. It is therefore no wonder that the subject is held with such derision in many people's minds. There is a very well-written argument entitled *A Mathematician's Lament* by Paul Lockhart, and it is a highly-recommended read if one were so-inclined to understand why education in this field is done so very wrongly. As we move forward with this text, try and keep everything said here in mind. This subject pays off in ways that we may not be aware of, but the payoff is there.

So welcome to Calculus. It'll be a ride, but these notes and exercises should help us as we take our journey through this subject. This chapter concerns itself with the concepts of change on extremely small scales, and codifies a concept we've seen before (when dealing with end behavior).

1.2 The Notion of a Limit

1.2.1 An Introduction to Problem Graphs

■ **Example 1.1** Let's begin with an example. Observe the graph in figure 1.1 below, and note the graph on the function $f(x)$ at the points a and b.

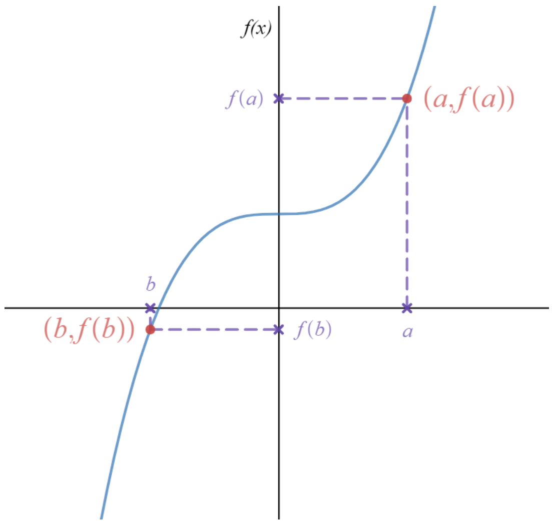

Figure 1.1: A graph of a function with points labeled.

Notice how both $f(a)$ and $f(b)$ are *defined*. That is, if we go to a on the x-axis and 'hop up' on the curve made by the function f, there is actually a point $(a, f(a))$ there. The question we want to ask in this chapter is... what if the point isn't defined there? ■

■ **Example 1.2** To investigate this further, let's take a look at a different function with a different point, except this time it's said to be *undefined* at the point $(b, f(b))$. We use an open circle to denote that a function is undefined at a particular point.

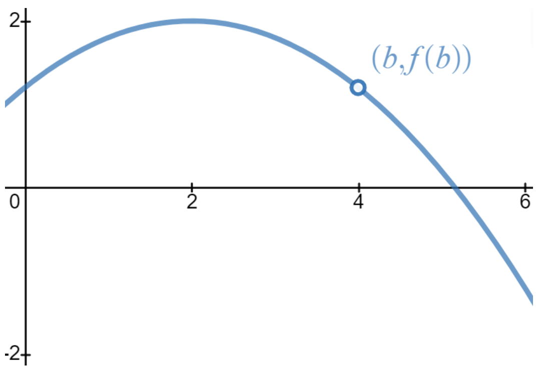

Figure 1.2: A graph of a parabola with a discontinuity at $x = 4$.

Notice how $b = 4$ in graph 1.2 above. So the function value $f(b) = f(4)$, but we know $f(4)$ is undefined, because there is a circle at that point. But doesn't it seem, just by looking at the graph, that $f(4) = 1$? The height of the circle seems to be about one unit above the x-axis, but we currently only have tools at our disposal to discuss where *points* are. We can't yet talk about where *holes* (or problems) in the graph are (or where points are *not*, depending on how you want to look at it). ■

Note 1.2.1 We can practice these ideas some more in the next example, but do note that some 'problems with the graph' are worse than others.

■ **Example 1.3** There is a terrible function below, and we can't help but notice that it's pretty jacked up at the points $x = -2$, $x = 2$, $x = 3$, $x = 4$, $x = 5$, and $x = 6$. We call these 'jacked-up' places *discontinuities*, and we'll talk more on them in the next section.

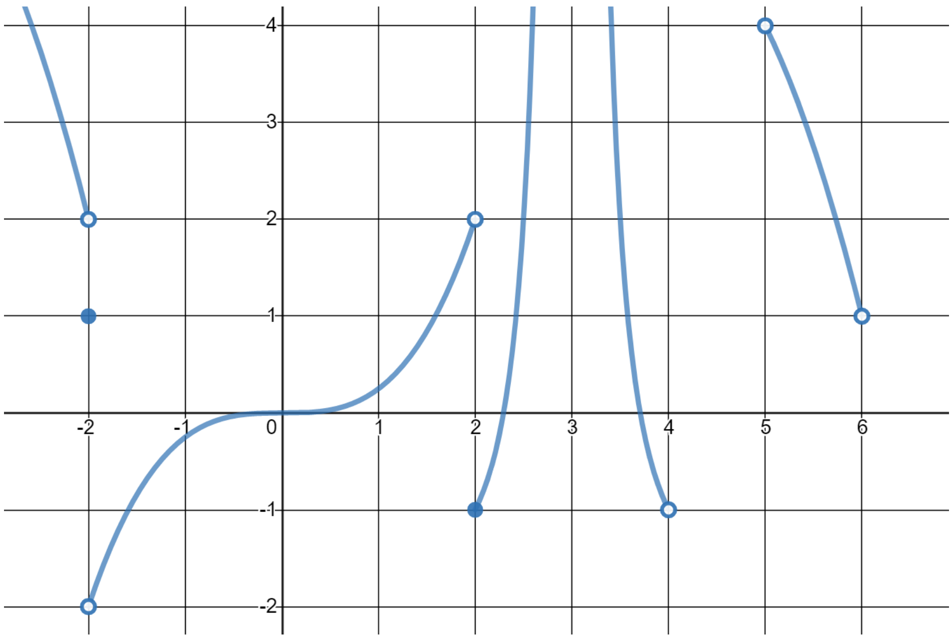

Figure 1.3: A rather jacked-up function.

So looking at the graph, we can deduce, for instance, that the graph is defined at $x = 0$, since $f(0) = 0$. We can also say that the graph is defined at $x = 2$, since $f(2) = -1$ (that is, the filled-in circle represents where the function is defined, while the circle still indicates where it isn't).

We also know that $f(4)$ is undefined, since there is no point on the vertical line $x = 4$ through which the graph of f passes. ■

Now it's your turn.

Exercise 1.1 Using the graph in the figure above, find the value of $f(-2)$. ■

Exercise 1.2 Using the graph in the figure above, find the value of $f(3)$. ■

Exercise 1.3 Using the graph in the figure above, estimate the value of $f(-1.5)$. ■

1.2.2 The Idea Behind Limits

It's all well-and-good to say what we can about the defined points, but so far, we lack the ability to talk about a function's behavior at an undefined point. Indeed, we'll never have the tools to talk about a function's behavior at a point at which it isn't defined because... *it isn't defined there.* By definition, it is impossible to speak of something if you don't have the words!

This would be the end of a very short book if it weren't for the ideas of Gottfried Wilhelm Leibniz and Sir Issac Newton in the late seventeenth and early eighteenth centuries. These two, the former a German and the latter an Englishmen, discovered Calculus separately. Their details are included in an appendix for your reading later.

For now, we can talk about the ingenious way in which both mathematicians tackled the issue. First, let's examine again the function from before, except zoomed in around the point $x = -2$. Note that the line $x = -2$ has been bolded so we can see it better.

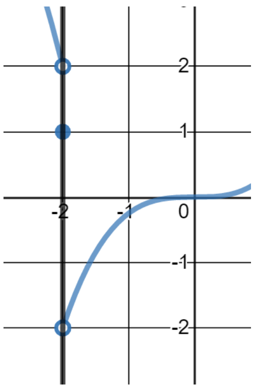

Figure 1.4: A rather jacked-up function returns!

■ **Example 1.4** Now, let's consider talking about the behavior at the point $x = -2$. It should be rather obvious that there are three points (two circles and one filled-in circle) on the line $x = -2$. It is helpful to think of *how we get* to these points. Imagine 'driving along' (or approaching, if you want to use the correct term,) the curve of f.

We move from left-to-right when reading words, and we do the same when reading graphs. So imagine you are approaching the line $x = -2$, which is bolded in the diagram. Clearly, we will 'get to' the hole at the top first, and the hole is located at the point $(-2, 2)$. Similarly, we could say we 'get to' the hole at the bottom located at $(-2, -2)$ if we approached from the right instead. ■

Definition 1.2.1 We say $x \to a^-$ to say that x approaches the point a from the left.

■ **Example 1.5** We can examine the behavior as x approaches -2 from the left (that is $x \to -2^-$) to talk about the hole at the point $(-2, 2)$. We can think about these values getting closer to -2 by starting at some arbitrary point on the number line to the left of 2 (in our case, we chose -2.1,) and getting closer. Getting closer... forever. This is the *notion* of a limit.

Mathematically, we notate the idea of $x \to -2^-$ as such:

That is, we choose a number to the left of $x = -2$, and get closer with each step. On a number line with a logarithmic scale, since each point on the line is arrived at multiplicatively:

■

Definition 1.2.2 We say $x \to a^+$ to say that x approaches the point a from the right.

■ **Example 1.6** We can examine the behavior as x approaches -2 from the right (that is $x \to -2^+$) to talk about the hole at the point $(-2, -2)$. We can think about these values getting closer to -2

-2.1

-2.01

-2.001

-2.0001

-2.00001

$\vdots$(continuing forever...)

$\downarrow$

-2

Figure 1.5: Approaching -2 from the left numerically.

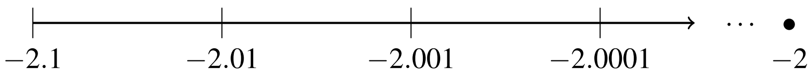

Figure 1.6: Approaching -2 from the left on a number line.

by starting at some arbitrary point on the number line to the right of 2 (in our case, we chose -1.9,) and getting closer.

Mathematically, we notate the idea of $x \to -2^+$ as such:

$x \to -2^+$

-1.9

-1.99

-1.999

-1.9999

-1.99999

$\vdots$(continuing forever...)

$\downarrow$

-2

Figure 1.7: Approaching -2 from the right numerically.

That is, we choose a number to the right of $x = -2$, and get closer with each step. On a number line with a logarithmic scale, since each point on the line is arrived at multiplicatively:

■

Either way, as we approach -2 from either the left or the right, the graph of f is clearly headed somewhere. We say as $x \to -2^-$, $f(x) \to 2$ and as $x \to -2^+$, $f(x) \to -2$. The figure is given one last time for your reference:

> **Note 1.2.2** Note that $f(-2) = 1$, since the graph is defined there by the given point.

1.2.3 Limit Notation

So we have now this idea of what limits do. Loosely, a limit can show us this idea of what a function 'would do' at a point, or what a function 'is going to' in the case of asymptotic breaks.

That's handy, for sure, but it is far too much writing to write out something that is such an easy idea. Fortunately, we have notation for that.

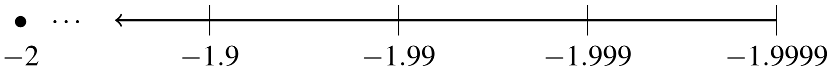

Figure 1.8: Approaching -2 from the right on a number line.

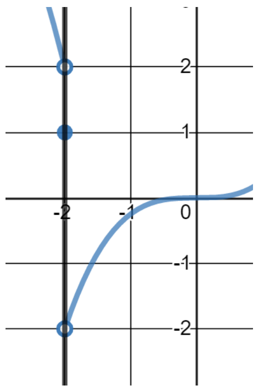

Figure 1.9: A rather jacked-up function is back again!

> **Note 1.2.3** For the following three definitions, we include the idea of L being either a real number or $\pm\infty$. This is loose terminology we'll nail down in a much later chapter.

The first defines the idea of a limit 'from the left.'

Definition 1.2.3 — The Left-handed Limit. Given $a \in \mathbb{R}$ and a function $f(x)$ we may write 'as $x \to a^-$, $f(x) \to L$' in *limit notation* as

$$\lim_{x \to a^-} f(x) = L.$$

This is read 'the limit as x goes to (or approaches) a from the left of $f(x)$ equals L.'

The second defines the idea of a limit 'from the right.'

Definition 1.2.4 — The Right-handed Limit. Given $a \in \mathbb{R}$ and a function $f(x)$ we may write 'as $x \to a^+$, $f(x) \to L$' in *limit notation* as

$$\lim_{x \to a^+} f(x) = L.$$

This is read 'the limit as x goes to (or approaches) a from the right of $f(x)$ equals L.'

Both of these are the same concepts we'd already discussed, just in a much more compact form. This begs the question, though... what if those two limits are approaching the same point? That is, what if they are equal? That is, more formally for some $a \in \mathbb{R}$,

Definition 1.2.5 — The Two-sided Property. $x \to a^-$ and $x \to a^+$ if and only if $x \to a$.

In this case, we simply use do not use the '$+$' or '$-$' signs. So if there is no sign after your a value, both the left- and right-handed limits happen at the same time.

Definition 1.2.6 — The Two-sided Limit. Given $a \in \mathbb{R}$ and a function $f(x)$ we may write 'as $x \to a$, $f(x) \to L$' in *limit notation* as

$$\lim_{x \to a} f(x) = L.$$

This is read 'the limit as x goes to (or approaches) a of $f(x)$ equals L.'

■ **Example 1.7** Let's take another look at our jacked-up function from before, and get used to the notation.

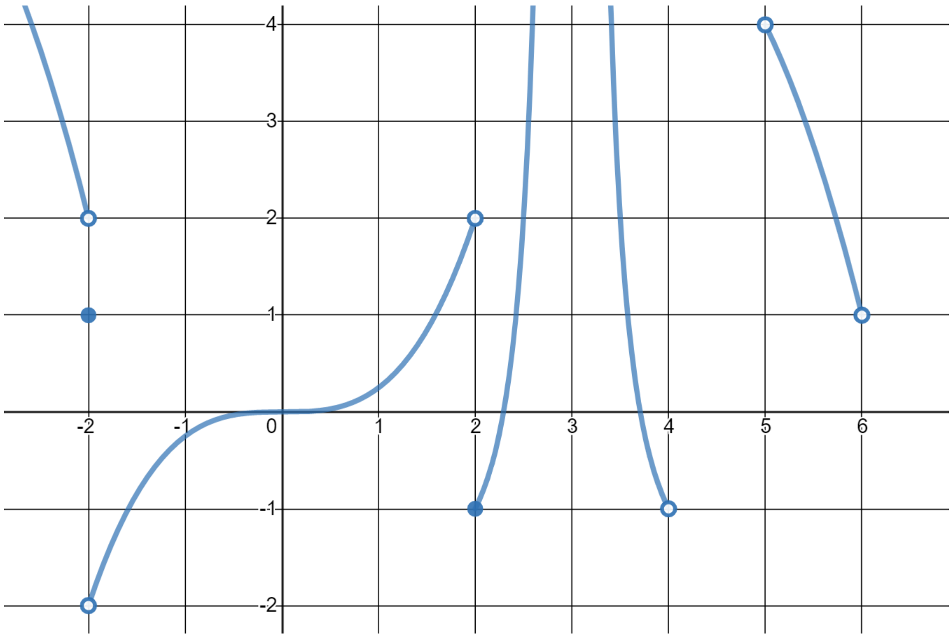

Figure 1.10: A rather jacked-up function.

Note 1.2.4 We know all the following are true:

1. $\lim_{x\to -2^-} f(x) = 2$
2. $\lim_{x\to -2^+} f(x) = -2$
3. $\lim_{x\to -2} f(x)$ does not exist
4. $\lim_{x\to 0} f(x) = 0$
5. $\lim_{x\to 3^+} f(x) = \infty$
6. $\lim_{x\to 4^-} f(x) = -1$

From this, we can ascertain that $\lim_{x\to -2} f(x)$ does not exist because $\lim_{x\to -2^-} f(x) = 2$ and $\lim_{x\to -2^+} f(x) = -2$. Since we know that $2 \neq -2$, we have that

$$\lim_{x\to -2^-} f(x) \neq \lim_{x\to -2^+} f(x).$$

Since the left- and right-handed limits do not equal the same number, the limit does not exist.

■

Exercise 1.4 Explain why $\lim_{x\to 0} f(x) = 0$ in Figure 1.10. ■

Exercise 1.5 Explain why $\lim_{x\to 3^+} f(x) = \infty$ in Figure 1.10. ■

Exercise 1.6 Explain why $\lim_{x\to 4.5^-} f(x)$ does not exist in Figure 1.10. ■

1.2.4 Limits of Symbolic Functions

So far, we've not explicitly defined $f(x)$ for any of our examples. Suppose we did that with a piecewise function as below.

Example 1.8

$$f(x) = \begin{cases} x+2 & x < -4 \\ -2 & -4 \leq x < 2 \\ x-2 & x > 2 \end{cases}$$

Note 1.2.5 Remember that a piecewise-defined function is simply a function that assumes a different definition depending on the input. In the above function, $f(x) = x+2$ as long as you are inputting a number less than -4. You have that $f(x) = -2$ if you have a number between -4 and 2, including -4 but not including 2. Finally, we have $f(x) = x-2$ if the input is strictly greater than 2.

Let's say in this problem we were asked to find $\lim\limits_{x \to -4^-} f(x)$. Since this is asking us to approach $a = -4$ from the left, the numbers as we approach in that direction must be less than -4. Therefore, we should use the 'first' piece of the graph. Then we have that

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} x+2.$$

We'll dive more into limit laws in a later section, but to evaluate $\lim\limits_{x \to -4^-} x+2$, we simply plug in -4 to $x+2$. So we obtain

$$\lim_{x \to -4^-} f(x) = \lim_{x \to -4^-} x+2 = (-4)+2 = -2.$$

■

Exercise 1.7 Given the piecewise function from the last example, evaluate $\lim\limits_{x \to -4^+} f(x)$. ■

Exercise 1.8 Using the solution to the last exercise and the solution from the last example, what can we conclude about the two-sided limit $\lim\limits_{x \to -4} f(x)$? Does it exist? Why or why not? ■

We will now graph the function

$$f(x) = \begin{cases} x+2 & x < -4 \\ -2 & -4 \leq x < 2 \\ x-2 & x > 2. \end{cases}$$

Exercise 1.9 Using the graph of the function in Figure 1.11, what can we conclude about the two-sided limit $\lim\limits_{x \to -4} f(x)$? Does it exist? Why or why not? What is it about the graph that tells us this information? ■

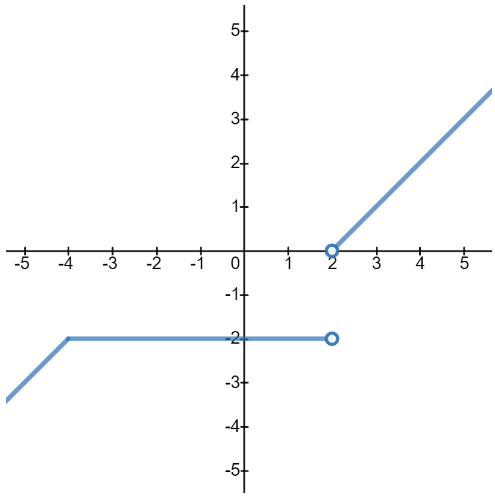

Figure 1.11: A graph of a piecewise-defined function.

Exercise 1.10 Using the same piecewise-defined function as before, evaluate $\lim_{x\to 2} f(x)$ numerically (that is, without using the graph). Use the graph to check your answer.

R Remember that you need to find both the left- and right-handed limits!

■

Exercise 1.11 Using the same piecewise-defined function as before, evaluate $\lim_{x\to 1} f(x)$ numerically. Use the graph to check your answer. ■

1.2.5 Limits and Tables

We're about to talk about weird limits, and for that we need functions that aren't continuous. If we're going to bring up continuity (the state of being continuous,) we may as well give an informal definition for now. The formal definition gets its own section!

Definition 1.2.7 — Informal Definition of Continuity. A function $f(x)$ is said to be continuous at the point $x = a$ if it doesn't do anything strange there. Strangeness at a point would be a hole in the graph, an asymptote, or simply not even existing at that point (like $\sqrt{x}$ for $x < 0$ defined on $\mathbb{R}$).

As mentioned, evaluating a limit has a set of rules that we'll discuss at the end of the chapter. For now, assume that the functions we want to evaluate are continuous (which is understandably frustrating, since the definition of continuity depends on a limit!) Since the functions of which we are taking limits are continuous, evaluating them is often as simple as 'plugging in' the number in the limit.

■ **Example 1.9** To find $\lim_{x\to 1^-} x+2$, we simply do

$$\lim_{x\to 1^-} x+2 = (1)+2 = 3,$$

because $x+2$ is a linear function, and all linear functions are continuous, a fact we will prove later. ■

Exercise 1.12 Find $\lim_{x \to -6^+} 1 - 3x$. ■

Exercise 1.13 Find $\lim_{x \to 6} 2x + 4$. ■

Note 1.2.6 Keep in mind that this method of 'plugging in' is something we have yet to show actually works! We are using this fact now because it greatly simplifies the learning process, but it also is like saying the punchline of a joke halfway through the joke.

So if 'plugging in' works to evaluate continuous functions, why do we even need the limit notation? A few seconds of thought will lead us to the obvious solution: not all functions are continuous everywhere.

■ **Example 1.10** Evaluate $\lim_{x \to 0^+} \frac{1}{x^2}$.

If we try the 'plugging in' method, which we'll call substituting from now on, we obtain

$$\lim_{x \to 0^+} \frac{1}{x^2} = \frac{1}{(0)^2} = \frac{1}{0},$$

which is clearly undefined. But this limit isn't undefined! Remember the central idea behind a limit is that we only get to the point we're going (0 in this case,) 'at *infinity*.' The idea of the limit, then, is that we never get to the $\frac{1}{0}$. Instead, we keep getting closer to that point forever and observe a behavior. For this, we need a table.

In this table, we're going to choose our x values based on the limit. In this case, $x \to 0^+$, meaning we need to choose a number just a bit bigger than 0 and get smaller towards 0. That is, we'll choose 1 because it's easy, and then get closer by jumping multiplicatively.

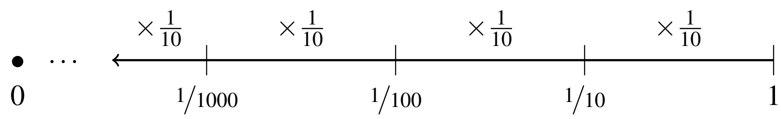

Figure 1.12: Approaching 0 from the right on a number line.

x	$f(x) = 1/x^2$
1	1
$1/10$	100
$1/100$	10000
$1/1000$	1000000

Note that we are plugging $1/10$ into $1/x^2$. This gives

$$\frac{1}{(1/10)^2} = 1 \cdot (10/1)^2 = 100.$$

The same holds for plugging in the other fractions.

So we can note that as we input values that get closer to 0 into our function, our output seems to be increasing without bound. It is clear that it does so *ad infinitum*, so we conclude by saying that

$$\lim_{x\to 0^+} \frac{1}{x^2} = \infty.$$

■

Note 1.2.7 The notation here varies depending on how particular the mathematician is. There are several people that work with such limits that would retire to their fainting couch if they dare saw us writing '$=\infty$' in print! They'd argue that nothing can equal infinity, but instead only approach it. For them, I present the definition that follows.

Definition 1.2.8 The function $f(x) = \pm\infty$ if and only if $\lim_{x\to a} f(x) \to \pm\infty$.

Now that we have that issue out of the way, let's walk through one more example before we try a few ourselves.

■ **Example 1.11** Evaluate $\lim_{x\to 1^-} \frac{1}{x-1}$.

So we'll notice that substitution doesn't work here, as $\frac{1}{(1)-1} = \frac{1}{0}$, which is obviously undefined. So it stands to reason that we'll have to use a table for this.

Since x is approaching 1 from the left, we need to start by choosing a number just a bit to the left of 1 on a number line. We'll note that 0 works well enough. Then we need to keep picking numbers that approach 1, so that all of them are less than 1. While you could try to use the sequence

$$\{0, {}^1\!/_5, {}^2\!/_5, {}^3\!/_5, {}^4\!/_5, \ldots\},$$

you'll notice that this sequence actually *gets to* 1 (and surpasses it,) which is something that never happens with limits! So instead we'll use the sequence

$$\{0, {}^{9}\!/_{10}, {}^{99}\!/_{100}, {}^{999}\!/_{1000}, {}^{9999}\!/_{10000}, \ldots\},$$

which you may recognize better in decimal form as

$$\{0, 0.9, 0.99, 0.999, 0.9999, \ldots\}.$$

The sequence looks something like the following on a number line:

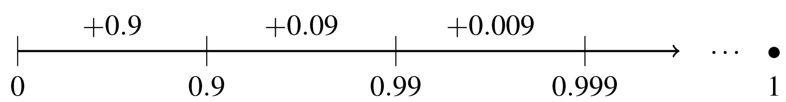

Figure 1.13: Approaching 1 from the left on a number line.

In tabular form, our choices show a clear trend.

x	$f(x)$
0	$\frac{1}{(0)-1} = -1$
0.9	$\frac{1}{(0.9)-1} = -10$
0.99	$\frac{1}{(0.99)-1} = -100$
0.999	$\frac{1}{(0.999)-1} = -1000$
0.9999	$\frac{1}{(0.9999)-1} = -10000$

Since $f(x) = \to -\infty$ as $x \to 1^-$, we can conclude that

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty. \tag{1.1}$$

Figure 1.14 shows a graph of $f(x)$. Notice how as $x \to 1^-$, the output approaches $-\infty$. Graphically, this is what Equation 1.1 means.

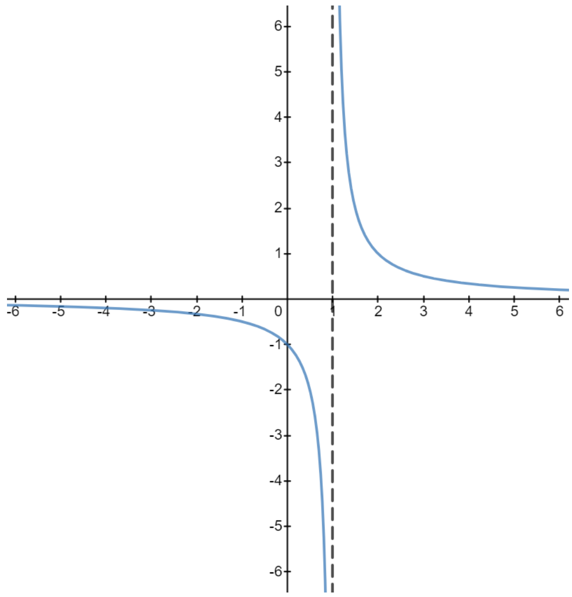

Figure 1.14: A graph of $f(x) = \frac{1}{x-1}$.

■

Note 1.2.8 — Making Good Choices for a Limit Table. Choosing the numbers for a limit table is more of an art than a science, but there are a few hard and fast rules. Let us assume that we are asked to evaluate the limit

$$\lim_{x \to 2} \frac{1}{x}.$$

Since this is a two-sided limit, we must find both

$$\lim_{x \to 2^-} \frac{1}{x} \quad \text{and} \quad \lim_{x \to 2^+} \frac{1}{x}.$$

We'll begin by finding $\lim_{x \to 2^-} \frac{1}{x}$. Clearly the limit evaluates to a value of $\frac{1}{2}$ (or 0.5 as a decimal), since $\frac{1}{x}$ is defined at $x = 2$, but for the purposes of practicing the art of zen and choosing points for a limit table, this will work well enough.

Since we are approaching the value $x = 2$ from the left, we must start with a number to the left of $x = 2$ on a number line. It should be clear that $x = 1$ is to the left of $x = 2$, so we can start with that value.

We then need to jump closer to the number, while staying to the left of it. Since the idea of a limit concerns the behavior close to the point in question ($x = 2$ in this case,) we jump far closer to $x = 2$, so $x = 1.9$ in this case. Then we jump even closer, say $x = 1.99$. We continue this until we've built our table.

$x \to 2^-$	$f(x) = 1/x$
1.1	$1/1 = 1$
1.9	$1/1.9 \approx 0.52632$
1.99	$1/1.99 \approx 0.50251$
1.999	$1/1.999 \approx 0.50025$
1.9999	$1/1.9999 \approx 0.50002$

This is enough to show that $\lim_{x\to 2^-} \frac{1}{x} = 0.5$ A similar approach works for approaching from the other direction.

$x \to 2^+$	$f(x) = 1/x$
2.1	$\frac{1}{2.1} \approx 0.47619$
2.01	$\frac{1}{2.01} \approx 0.49751$
2.001	$\frac{1}{2.001} \approx 0.49975$
2.0001	$\frac{1}{2.0001} \approx 0.49998$
2.0001	$\frac{1}{2.00001} \approx 0.499998$

This shows that $\lim_{x\to 2^+} \frac{1}{x} = 0.5$ Since both the left- and right-handed limits approach the same point, we can conclude that

$$\lim_{x\to 2} \frac{1}{x} = 0.5.$$

Exercise 1.14 Evaluate $\lim_{x\to 3} \dfrac{1}{3-x}$ using a table. ■

Exercise 1.15 Evaluate $\lim_{x\to 0} \dfrac{1}{|x|}$ using a table. ■

Exercise 1.16 Evaluate $\lim_{x\to 2} \dfrac{1}{x^2-2x}$ using a table. ■

Exercise 1.17 Evaluate $\lim_{x\to\pi} \csc x$ using a table and a calculator. ■

1.3 The Precise Definition of a Limit

1.3.1 The Intuition of the Definition

So far, we've spoke of limits as this idea of talking about 'bad behavior' at a point by talking about sequences of points *approaching* the 'bad' point in question. This is a useful notion, but it is not at all rigorous. Since a formal basis is ideal for any mathematical subject, let us now begin to look at the precise definition of a limit. Consider Figure 1.15.

This is a graph of a function, specifically $f(x) = x^{\frac{6}{5}}\sin(2x)$. Let's set $f(a) = L$, since it's

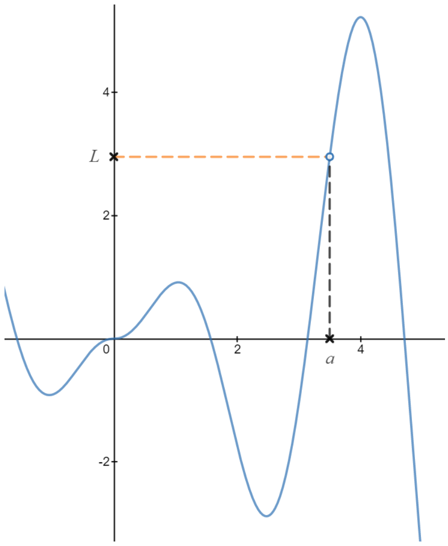

Figure 1.15: A function with a discontinuity at $x = a$.

the limit we wish to discuss. Notice the discontinuity at the point (a, L). Now let us construct a δ-neighborhood about a.

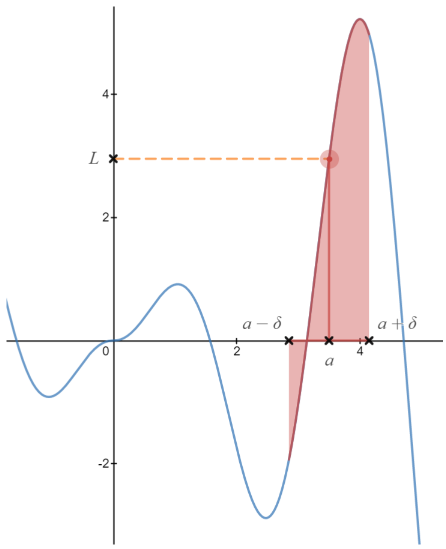

Figure 1.16: A neighborhood of δ about $x = a$.

Note 1.3.1 — Delta and Epsilon. The Greek letters ε (epsilon) and δ (delta) are real, positive numbers used in these sorts of proofs. Going forward, try and think of ε as a *very small* number. It will help with your intuition of what is going on.

Definition 1.3.1 — Neighborhood. A neighborhood of d about the point $x = a$ is the interval $(a - d, a + d)$. That is, it's the set of points distance d from the center point a.

So now that we have some horizontal distances represented in one neighborhood, let us construct an ε-neighborhood about L in Figure 1.17.

Notice how the red tracing along the curve of $f(x)$ is not contained inside the orange band

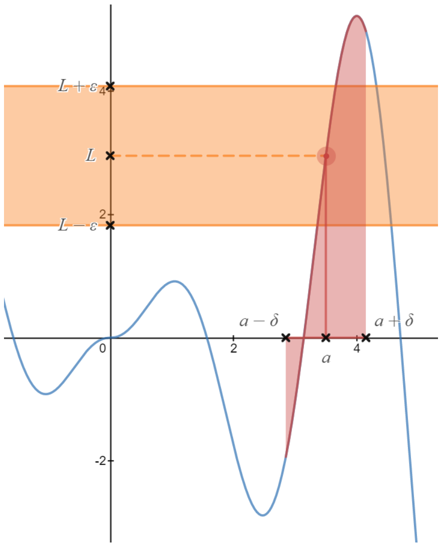

Figure 1.17: A neighborhood of ε about $f(x) = L$.

representing the ε-neighborhood about L. The idea behind the precise definition of a limit is fairly straightforward: if we're given an ε-neighborhood, we can choose δ so that our δ-neighborhood fits inside the band created by the ε-neighborhood when the δ-neighborhood is traced along the curve. If this is true for any $\varepsilon > 0$ that is chosen, then we'll 'squeeze' in on the limit. We'll formally define this idea now.

Definition 1.3.2 — The Precise Definition of a Limit. Given $\varepsilon, \delta, a, L \in \mathbb{R}$,

$$(\forall \varepsilon > 0)(\exists \delta > 0)(|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

This is read 'For all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|x-a| < \delta$, then $|f(x) - L| < \varepsilon$. This is what it means when we use limit notation to say

$$\lim_{x \to a} f(x) = L.$$

■ **Example 1.12** Let's continue with the same function as before, and make this into a game of sorts. We have control over the δ, and our opponent has control over the ε. We both take turns choosing values that are smaller so that we can fit our neighborhood (traced along the curve of the function) inside our opponent's. So then we'll choose a smaller δ than the one we chose in Figure 1.17. Then Figure 1.18 shows that we now fit clearly inside our opponent's ε neighborhood!

Our opponent then chooses a smaller ε, shown in Figure 1.19, and we're right back outside the ε-neighborhood again.

Note 1.3.2 Note that when we say we're 'outside' the ε-neighborhood, we mean that the trace of our δ-neighborhood (the red part of the blue curve) has parts outside of the ε-neighborhood (the orange band).

We are, of course, welcome to simply choose a smaller δ value, given our opponent's new choice for ε, which we will do in Figure 1.20.

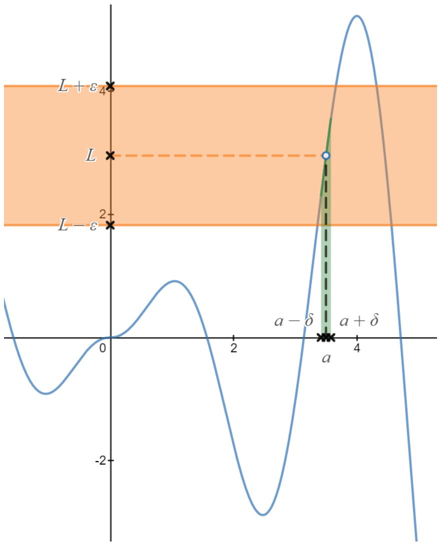

Figure 1.18: A correct choice for δ given ε.

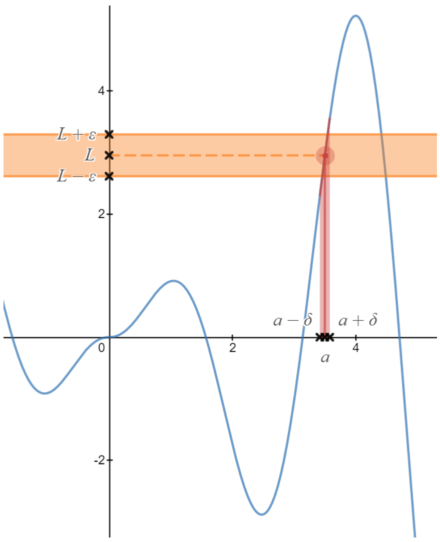

Figure 1.19: Now ε is too small. Need to adjust δ again.

Note 1.3.3 It should be obvious at this point that this process can continue *ad infinitum* (which is the Latin phrase roughly meaning forever). This means that the limit exists. We'll give an example in a moment in which this process does not work, and therefore the limit would not exist for that example.

■

1.3.2 When the Limit Breaks

It's worth noting that it's not always possible to find a δ so that the trace of the δ-neighborhood fits along the curve of $f(x)$ inside the given ε-neighborhood. Consider the graph shown in Figure 1.21.

Exercise 1.18 Explain how it is impossible to fit the trace of a δ-neighborhood inside the ε-neighborhood shown in Figure 1.21. ■

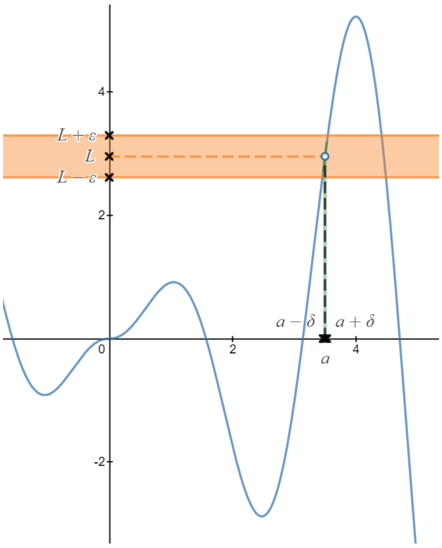

Figure 1.20: A correct choice for the new δ given the newer, smaller ε.

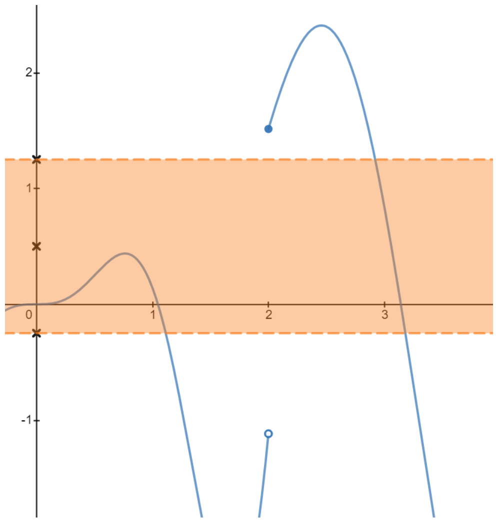

Figure 1.21: Here it is impossible to choose a δ given this ε.

1.3.3 The Epsilon-Delta Proof

While being one of the more complicated *looking* topics discussed in a Calculus class, this isn't actually terrible, mostly because there's so much structure to it. To wit, the process to writing a proof for a limit is rather simple.

Note 1.3.4 — The Process for Writing an ε-δ Proof.. We'll note that the process for these proofs is always the same.

1. Identify your a and L values from the problem.
2. Identify the function $f(x)$ from the problem.
3. Assume $|x-a| < \delta$, filling in your a value.
4. Knowing $|f(x)-L| < \varepsilon$, fill in $f(x)+L$.
5. Work backwards from $|f(x)-L| < \varepsilon$ to find an equation for δ in terms of ε.
6. Knowing this 'roadmap' for your proof, write the proof.

This is perhaps a bit esoteric without an example, so let's try this.

■ **Example 1.13** Prove $\lim_{x\to 2} x+3=5$.

Recall when we said earlier that we could just 'plug in' $a=2$ to $x+3$ to evaluate the limit? This was correct, but now we'll do this with rigor, and show why it works for all linear functions in a later example. Using our process from above, we first notice that $a=2$, $L=5$ and $f(x)=x+3$. Then we must show, using those values in the definition,

$$(\forall\varepsilon>0)(\exists\delta>0)(|x-2|<\delta\Rightarrow|(x+3)-(5)|<\varepsilon).$$

Then we are beginning with $|x-2|<\delta$ and must end with $|(x+3)-(5)|<\varepsilon$. Simply doing the math in the ε inequality gives us that $|x-2|<\varepsilon$. It should be pretty clear that the only thing we need do to make these two inequalities equal is to let $\delta=\varepsilon$. It isn't usually this simple, but it's a good first step.

Note 1.3.5 The scratch work for this problem would look something like the following:

$$\begin{aligned} |x-a| &< \delta \\ |x-2| &< \delta \\ |x-2| &< \varepsilon \\ |(x+3)-5| &< \varepsilon \\ |f(x)-L| &< \varepsilon \end{aligned}$$

You'd write the $|x-2|<\delta$ at the top and the $|(x+3)-(5)|<\varepsilon$ at the bottom and work towards the middle to find δ in terms of ε by getting the left side of the inequality to look more like the topmost line as you work up from the bottom.

The proof is pretty simple as well, and follows the example.

Proof. Let $\varepsilon>0$. Choose $\delta=\varepsilon$. Then

$$|f(x)-L|=|(x+3)-5|=|x-2|<\delta=\varepsilon.$$

Then by the definition of a limit,

$$\lim_{x\to 2} x+3=5.$$

■

■

Exercise 1.19 Prove $\lim_{x\to -4} x+5=1$. ■

■ **Example 1.14** Prove $\lim_{x\to 3} 2x+1=7$.

First, we'll identify $a=3$, $L=7$, and $f(x)=2x+1$. Filling into the definition, we must show

$$(\forall\varepsilon>0)(\exists\delta>0)(|x-3|<\delta\Rightarrow|(2x+1)-(7)|<\varepsilon).$$

So we'll begin with what we know ($|x-a|<\delta$) and get to what we need to show ($|f(x)-L|<\varepsilon$). That is, in our scratch work, we'll have

$$\begin{array}{rcl}|x-a| & < & \delta \\ |x-3| & < & \delta \\ & \vdots & \\ |(2x+1)-(7)| & < & \varepsilon \\ |f(x)-L| & < & \varepsilon.\end{array}$$

So it should be pretty clear that we are working up from the bottom, trying to make the left side (which is currently $|(2x+1)-(7)|$) look like $|x-3|$. If we can do this, then we must only 'match up' our right sides to get our δ in terms of ε.

A It may even be smarter to think of δ as a function of ε. That is $\delta(\varepsilon)$.

Regardless, working from the bottom up, we fill out our scratch work further.

$$\begin{array}{rcl}|x-a| & < & \delta \\ |x-3| & < & \delta \\ & \vdots & \\ |2(x-3)| & < & \varepsilon \\ |2x-6| & < & \varepsilon \\ |(2x+1)-(7)| & < & \varepsilon \\ |f(x)-L| & < & \varepsilon.\end{array}$$

Notice how we factored out the 2 in the expression and obtained the $x-3$ term we were looking for. *This will always happen.* Further, we need to recall something about absolute value.

Definition 1.3.3 — Factoring in Absolute Value. Given numbers $a, b \in \mathbb{R}$,

$$|a \cdot b| = |a| \cdot |b|.$$

Then we may finish by factoring $|2(x-3)| = |2||x-3| = 2|x-3|$ and dividing over the 2 to obtain

$$\begin{array}{rcl}|x-a| & < & \delta \\ |x-3| & < & \delta \\ |(x-3)| & < & \varepsilon/2 \\ 2|(x-3)| & < & \varepsilon \\ |2||(x-3)| & < & \varepsilon \\ |2(x-3)| & < & \varepsilon \\ |2x-6| & < & \varepsilon \\ |(2x+1)-(7)| & < & \varepsilon \\ |f(x)-L| & < & \varepsilon.\end{array}$$

Then $\delta = \frac{\varepsilon}{2}$. This is all the information we need. We can just fill in our proof and we are done.

Proof. Let $\varepsilon > 0$. Choose $\delta = \varepsilon/2$ such that $|x-3| < \delta$. Then

$$|f(x) - L| = |(2x+1) - 7| = |2x-6| = |2||x-3| = 2|x-3| < 2\delta = 2\frac{\varepsilon}{2} = \varepsilon.$$

Then we have shown that

$$\lim_{x\to 3} 2x+1 = 7.$$

■

Note 1.3.6 It's pretty common for the inequality step of the proof to be the hardest to grasp. But note that $2|x-3| =< 2\delta$ because $|x-3| < \delta$. We simply multiplied each side by a factor of 2.

■

Exercise 1.20 Prove $\lim\limits_{x\to -3} 3x+5 = -4$. ■

Exercise 1.21 Prove $\lim\limits_{x\to -5} 5 - \frac{3}{5}x = 8$. ■

Exercise 1.22 Prove $\lim\limits_{x\to a}(mx+b) = ma+b$ for all $a, b, m \in \mathbb{R}$ and tell what this means. ■

1.3.4 Quadratic Limits

So far, we've spoke only of limits of linear functions. From a teaching perspective, this is good, since lines are one of the simplest non-constant functions in all of mathematics, and their proofs are pretty straightforward. But what if we need to work with the limits of a quadratic? The good news is that the process is exactly the same, the bad news is there's an extra step. First, let's recall a fact about absolute value.

Definition 1.3.4 — The Compound Definition of Absolute Value. If $a > 0$ and $|x| < a$, then

$$-a < x < a.$$

■ **Example 1.15** Prove $\lim\limits_{x\to 2} x^2 - 3 = 1$.

To begin this process, we'll set up our scratch work the same way as always.

$$\begin{array}{rcl} |x-a| & < & \delta \\ |x-2| & < & \delta \\ & \vdots & \\ |(x^2-3)-(1)| & < & \varepsilon \\ |f(x)-L| & < & \varepsilon. \end{array}$$

Working from the bottom up, we can simplify the $|(x^2-3)-(1)|$ expression to $|x^2-4|$. From there, we factor to obtain $|(x-2)(x+2)|$. This brings us to our next little trick we need to recall.

Note 1.3.7 — Factoring Polynomials in Absolute Value. If the polynomial $p(x)$ factors into the terms $w(x)$ and $u(x)$, then

$$|p(x)| = |w(x)u(x)| = |w(x)||u(x)|.$$

Then we can write $|(x-2)(x+2)|$ as $|x-2||x+2|$. This means our scratch work should be looking pretty good.

$$\begin{aligned}
|x-a| &< \delta \\
|x-2| &< \delta \\
&\vdots \\
|x-2||x+2| &< \varepsilon \\
|(x-2)(x+2)| &< \varepsilon \\
|x^2-4| &< \varepsilon \\
|(x^2-3)-(1)| &< \varepsilon \\
|f(x)-L| &< \varepsilon.
\end{aligned}$$

Since we're trying to get $|x-2|$ by itself, intuition would have us divide by $|x+2|$ We can do that, which brings us to a bit of a conundrum.

$$\begin{aligned}
|x-a| &< \delta \\
|x-2| &< \delta \\
|x-2| &< \tfrac{\varepsilon}{|x+2|} \\
|x-2||x+2| &< \varepsilon \\
|(x-2)(x+2)| &< \varepsilon \\
|x^2-4| &< \varepsilon \\
|(x^2-3)-(1)| &< \varepsilon \\
|f(x)-L| &< \varepsilon.
\end{aligned}$$

If we were to choose $\delta = \frac{\varepsilon}{|x+2|}$ here, we'd be making a rather large error. Remember that the δ we choose must be a function of ε *only*. So we need to place some sort of bound on the $|x+2|$ term.

In order to do that, let's just bound $\delta \leq 1$. We can do this, because the whole point of the game is to get δ small anyway, and it's going to be smaller than 1 at some point. We chose 1 somewhat arbitrarily, any positive number would have worked. It's just that 1 is pretty easy to work with.

So now that we have bounded our $\delta \leq 1$, we recall that $|x-2| < \delta$. Combining these two is the same as saying that $|x-2| < 1$. Using the note on the compound definition of the absolute value from earlier, we can see the following:

$$\begin{array}{rcccl}
|x-2| & < & 1 & & \\
-1 & < & x-2 & < & 1 \\
-1+(4) & < & x-2+(4) & < & 1+(4) \\
3 & < & x+2 & < & 5.
\end{array}$$

We began with our $|x-a| = |x-2|$ term (that is, the term we knew about). We turned that term into the term we were dividing by (that is, the $|x+2|$ term) by adding 4 to each part of the inequality. Then we know the biggest $x+2$ can get is 5, so we can write 5 instead of $x+2$ in the scratch work. We'll call this part of the problem the *bounding work*.

$$\begin{aligned}
|x-a| &< \delta \\
|x-2| &< \delta \\
|x-2| &< \tfrac{\varepsilon}{5} \\
|x-2| &< \tfrac{\varepsilon}{|x+2|} \\
|x-2||x+2| &< \varepsilon \\
|(x-2)(x+2)| &< \varepsilon \\
|x^2-4| &< \varepsilon \\
|(x^2-3)-(1)| &< \varepsilon \\
|f(x)-L| &< \varepsilon.
\end{aligned}$$

Then we can just choose $\delta = \frac{\varepsilon}{5}$ and we're ready to go, right? Almost. What if the ε is so large that our restriction of $\delta \leq 1$ matters? The answer is super easy, barely an inconvenience.

Definition 1.3.5 The minimum of a set of values $\{a_1, a_2, \ldots, a_n\}$ is denoted $\min\{a_1, a_2, \ldots, a_n\}$, and is defined to be the smallest element.

So we must simply choose $\delta = \min\left\{\frac{\varepsilon}{5}, 1\right\}$ and everything works out! The proof follows, but looks remarkably the same, save for the extra bounding work.

Proof. Let $\varepsilon > 0$. Bound $\delta \leq 1$ so that $|x-2| < \delta$. Choose $\delta = \min\{1, \varepsilon/5\}$. Then

$$|(x^2-3)-1| = |x^2-4| = |x-2||x+2| < \delta \cdot (5) = \varepsilon/5 = \varepsilon.$$

Then

$$\lim_{x\to 2} x^2 - 3 = 1.$$

■

Note 1.3.8 In the inequality step in the above proof, we say $|x-2||x+2| < \delta \cdot (5)$. This is because both $|x-2| < \delta$ and $|x+2| < 5$ by our bounding work.

■

Then we can restate our notes on the process to include these new steps.

Note 1.3.9 — The Process for Writing an ε-δ Proof. We'll note that the process for these proofs is always the same.

1. Identify your a and L values from the problem.
2. Identify the function $f(x)$ from the problem.
3. Assume $|x-a| < \delta$, filling in your a value.
4. Knowing $|f(x)-L| < \varepsilon$, fill in $f(x)-L$.
5. Work backwards from $|f(x)-L| < \varepsilon$ to find an equation for δ in terms of ε.
6. During this process, you'll need to find a bound on a function of x you have divided ε by.
7. In order to do this, bound $\delta \leq 1$ and expand it to a compound inequality.

8. Get your $|x-a|$ to look like the term with which you are dividing ε.
9. Find your δ from this information.
10. Knowing this 'roadmap' for your proof, write the proof.

Exercise 1.23 Prove $\lim_{x\to 1} x^2+4x+1=6$. ■

Exercise 1.24 Prove $\lim_{x\to 1} x^2+4x+3=8$ ■

Exercise 1.25 Prove $\lim_{x\to -3} 2x^2+6x+2=2$. ■

Exercise 1.26 Prove $\lim_{x\to 0} 3x^2-2x-1=-1$. ■

Exercise 1.27 Prove $\lim_{x\to 0} 16-x^2=16$. ■

Exercise 1.28 Prove $\lim_{x\to 4} x^2-\frac{x+4}{4}=14$. ■

Exercise 1.29 Prove $\lim_{x\to 1} \frac{1}{3}x+\frac{2}{3}=1$. ■

1.4 Limit Laws

We have both informally and formally defined the idea of a limit, and we've hand-waved the idea of substitution as long as we have this vague notion of continuity (which we have yet to really define rigorously). We do not yet know how to actually work with limits. To rectify this, we are going to define eight laws for limits, which are the rules for how these work. We will begin with the two most basic.

Definition 1.4.1 — Constant Law. Let $a, c \in \mathbb{R}$. Then

$$\lim_{x \to a} c = c.$$

Proof. Let $\varepsilon > 0$. Then

$$|f(x) - L| = |(c) - (c)| = |0| = 0 < \varepsilon,$$

thus proving that

$$\lim_{x \to a} c = c.$$

■

R Note that there was no need to even choose a value for delta in the proof, since the function $f(x) = c$ for $c \in \mathbb{R}$ will always be in an ε-neighborhood about L. That is, no matter your choice for δ, the entirety of $f(x)$ will lie inside the interval $(L - \varepsilon, L + \varepsilon)$.

Definition 1.4.2 — Substitution Law. Let $a, c \in \mathbb{R}$. Then

$$\lim_{x \to a} x = a.$$

Proof. Let $\varepsilon > 0$. Let $\delta = \varepsilon$ so that $|x - a| < \delta$. Then

$$|f(x) - L| = |x - a| < \delta = \varepsilon.$$

Therefore

$$\lim_{x \to a} x = a.$$

■

These two definitions provide the 'bedrock' for our use of these laws. The Constant Law states that the limit of a constant is a constant. The Substitution Law states that if we are taking the limit as x approaches a of only the identity function x, then we will obtain a. Our objective is going to be to get parts our limits to look like one of the left sides of these two definitions so we can substitute the value that does not contain a limit.

Definition 1.4.3 — Factor Law. Let $a, c \in \mathbb{R}$ and f be a function with a limit at a. Then

$$\lim_{x \to a} (c \cdot f(x)) = c \cdot \lim_{x \to a} f(x).$$

The Factor Law states that we can factor out constants from our limits. It's good practice to factor out constants, as we'll be doing quite a bit of that as we move forward into derivatives in the next chapter.

■ **Example 1.16** Show $\lim_{x \to 3} 2x = 6$ using limit laws.

This is easy enough since we only have the three laws so far.

$$\begin{aligned} \lim_{x \to 3} 2x &= 2 \lim_{x \to 3} x \\ &= 2(3) \\ &= 6 \end{aligned}$$

We used the Factor Law followed by the Substitution Law to find this limit. ■

Definition 1.4.4 — Sum and Difference Law. Let $a, K, L \in \mathbb{R}$ and f be a function with a limit of L at a and g be a function with a limit of K at a. Then

$$\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm K.$$

The Sum and Difference Laws state that the limit of a sum or difference is the sum or difference of the limits. That is, you can 'break apart' parts of a limit across both addition and subtraction operators. We'll find as we go through these laws that this 'breaking apart' property holds for pretty much any operation. Actually, limits usually behave exactly how we'd hope they would, something rare for mathematics!

■ **Example 1.17** Find $\lim_{x \to 4} 3x + 1$ using limit laws. Justify each step.

$$\begin{aligned} \lim_{x \to 4} 3x + 1 &= \lim_{x \to 4} 3x + \lim_{x \to 4} 1 && : \text{Addition} \\ &= 3 \lim_{x \to 4} x + (1) && : \text{Factor \& Constant} \\ &= 3(4) + (1) && : \text{Substitution} \\ &= 13 \end{aligned}$$

■

Exercise 1.30 Find $\lim_{x \to 0} 2 - (3x - 3)$ using limit laws. Justify each step. ■

We now define the rest of the laws, with more complicated examples to follow.

Definition 1.4.5 — Multiplication Law. Let $a, K, L \in \mathbb{R}$ and f be a function with a limit of L at a and g be a function with a limit of K at a. Then

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) = L \cdot K.$$

Definition 1.4.6 — Quotient Law. Let $a, K, L \in \mathbb{R}$ and f be a function with a limit of L at a and g be a function with a limit of $K \neq 0$ at a. Then

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right]=\frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}=\frac{L}{K}.$$

Definition 1.4.7 — Power Law. Let $a,n,K,L\in\mathbb{R}$ and f be a function with a limit of L at a. Then

$$\lim_{x\to a}[f(x)]^n=\left[\lim_{x\to a} f(x)\right]^n=L^n.$$

Definition 1.4.8 — Root Law. Let $a,n,K,L\in\mathbb{R}$ and f be a function with a limit of L at a. Then

$$\lim_{x\to a}\left[\sqrt[n]{f(x)}\right]=\sqrt[n]{\lim_{x\to a} f(x)}.$$

Note 1.4.1 These last four laws all follow the same basic rule. The limit can be taken over any given operation, provided the limit exists. To state it point-by-point,

1. The limit of a product is the product of the limits.
2. The limit of a quotient is the quotient of the limits.
3. The limit of a power is the power of the limit.
4. The limit of a root is the root of the limit.

R We have stated the Root Law, but it is really just a corollary of the Power Law, since $\sqrt[n]{x^m}=x^{\frac{m}{n}}$.

Now we'll attempt a complicated example that uses most of the laws in one problem!

R Do not forget to show each step when doing these sorts of problems. Simplify numbers as soon as you can, slowly inching the limits further inside the operations until you can invoke either the Substitution or Constant Laws. We will be finished when we have a number as an answer!

■ **Example 1.18** Find $\lim\limits_{x\to 1} x\sqrt{\dfrac{x-x^3}{x+1}}$ using limit laws. Justify each step.

$$\begin{aligned}
\lim_{x\to 1} x\sqrt{\frac{x-x^3}{x+1}} &= \lim_{x\to 1} x \cdot \lim_{x\to 1}\sqrt{\frac{x-x^3}{x+1}} && : \text{ Multiplication} \\
&= (1)\cdot\sqrt{\lim_{x\to 1}\frac{x-x^3}{x+1}} && : \text{ Substitution \& Root} \\
&= \sqrt{\frac{\lim_{x\to 1}(x-x^3)}{\lim_{x\to 1}(x+1)}} && : \text{ Quotient} \\
&= \sqrt{\frac{\lim_{x\to 1} x - \lim_{x\to 1} x^3}{\lim_{x\to 1} x + \lim_{x\to 1} 1}} && : \text{ Difference \& Sum} \\
&= \sqrt{\frac{(1)-\left(\lim_{x\to 1} x\right)^3}{(1)+(1)}} && : \text{ Substitution, Constant, \& Power} \\
&= \sqrt{\frac{(1)-(1)^3}{2}} && : \text{ Substitution} \\
&= \sqrt{\tfrac{0}{2}} \\
&= 0
\end{aligned}$$

■

Notice how we invoked the Substitution and Constant Laws as soon as we could, and did the arithmetic as soon as possible as well. This method relies on a lot of 'bookkeeping,' which is the math-esque term for 'meticulous, methodical, and boring.' Yet without the bookkeeping, we'd have no idea how we arrived at our answer! It's important that we understand exactly how the limit operator works, as it is the linchpin for the rest of our studies in this class.

Exercise 1.31 Find $\lim_{x\to 3}\sqrt{\frac{\sqrt{x+1}}{x+1}+x}$ using limit laws. Justify each step. ■

To wrap this section up, we'll restate all the limit laws in one easy-to-reference spot.

Definition 1.4.9 — The Limit Laws. Let $a, K, L \in \mathbb{R}$ and f be a function with a limit of L at a and g be a function with a limit of K at a (with $K \neq 0$ in law 6). Then the following are true:

1. $\lim_{x\to a} c = c$.
2. $\lim_{x\to a} x = a$.
3. $\lim_{x\to a}(c\cdot f(x)) = c\cdot\lim_{x\to a} f(x)$.
4. $\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm K$.
5. $\lim_{x\to a}[f(x)\cdot g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = L\cdot K$.
6. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{K}$.
7. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n$.
8. $\lim_{x\to a}\left[\sqrt[n]{f(x)}\right] = \sqrt[n]{\lim_{x\to a} f(x)}$.

Exercise 1.32 Use the limit laws to find $\lim_{x\to -3}(2x+2)$. Justify each step. ■

Exercise 1.33 Use the limit laws to find $\lim_{x\to\pi}\left(\frac{2}{\pi}x^2\right)$. Justify each step. ■

Exercise 1.34 Use the limit laws to find $\lim_{x\to 1}(x^2+4x+1)$. Justify each step. ■

Exercise 1.35 Use the limit laws to find $\lim_{x\to 4}\left(x\cdot\frac{x^2-1}{\sqrt{8-x}}\right)$. Justify each step. ■

Exercise 1.36 Find $\lim_{x\to -4} x^2\sqrt{\frac{1+3x-x^3}{4-\sqrt{-x}}}$ using limit laws. Justify each step. ■

Exercise 1.37 Use the limit laws to find $\lim_{x\to -1}\left(\sqrt[4]{-x}\cdot\frac{\sqrt[3]{x^2+x^3}}{\sqrt{3-x}}\right)$. Justify each step. ■

1.5 Finding Limits

There are some limits that resist our attempts to simply substitute the value being approached into it. So while a nice limit like

$$\lim_{x\to 3}(x^2+2x+1) = (3)^2+2(3)+1 = 16$$

works out for us, something like

$$\lim_{x\to 4}\frac{x-4}{x^2-16} = \frac{(4)-4}{(4)^2-16} = \frac{0}{0}$$

definitely does not work out nicely. In fact, what we get as an 'answer' is called an INDETERMINATE FORM and is something we'll talk about in more detail way later in the text. For now, just know it means there's more work to do before you get an answer. We'll go through this and see the trick is just basic factoring.

■ **Example 1.19** Evaluate $\lim_{x\to 4}\frac{x-4}{x^2-16}$.

This problem is as simple as noting that $x^2-16 = (x-4)(x+4)$, and therefore

$$\frac{x-4}{x^2-16} = \frac{x-4}{(x-4)(x+4)} = \frac{1}{x+4}$$

for $x \neq 4$. Since in our limit, $x \to 4$ this $x \neq 4$ restriction need not apply, and thus

$$\lim_{x\to 4}\frac{x-4}{x^2-16} = \frac{1}{x+4} = \frac{1}{(4)+4} = \frac{1}{8}$$

■

Exercise 1.38 Evaluate $\lim_{x\to -2}\frac{x+2}{x^2-4}$. ■

Exercise 1.39 Evaluate $\lim_{x\to -3}\frac{x^2-x-12}{x^2+4x+3}$. ■

Another kind of problem that can arise when computing particularly troublesome limits involves square roots. We will investigate this in the next example.

■ **Example 1.20** Find $\lim_{x\to 1}\frac{x+\sqrt{5-4x}}{1-x}$.

We note that simply evaluating the limit at $x=1$ gives a division by zero. Thus, we must do some additional work in order to simplify the function we are taking the limit of before we bother taking the limit. Here and elsewhere, it may be smart to work with just the function to simplify it, rather than carrying the limit through each step. Just remember to put it back before you evaluate!

Our problem in this example is that we are dividing by zero, so we need to somehow cancel the $1-x$ term in the denominator. Since the numerator is the only other thing we have to work with, we will make use of the difference of squares to get rid of the square root bit on the top. Namely, we will multiply both the numerator and denominator by the conjugate of the numerator. Since the numerator is $x+\sqrt{5-4x}$, its conjugate is $x-\sqrt{5-4x}$ and we therefore have that

$$\begin{aligned}\frac{x+\sqrt{5-4x}}{1-x} &= \frac{x+\sqrt{5-4x}}{1-x}\frac{x-\sqrt{5-4x}}{x-\sqrt{5-4x}}\\ &= \frac{(x+\sqrt{5-4x})(x-\sqrt{5-4x})}{(1-x)(x-\sqrt{5-4x})}\\ &= \frac{x^2-(5-4x)}{(1-x)(x-\sqrt{5-4x})}\end{aligned}$$

Your previous experience in mathematics might tell you to to multiply out the denominator as well. In this case, it would not be helpful. Remember that we are trying to cancel the $1-x$ term, so we'd lose it if we were to multiply it out! We continue by distributing through the negative on the numerator and rearranging things to get that

$$\begin{aligned}\frac{x^2-(5-4x)}{(1-x)(x-\sqrt{5-4x})} &= \frac{x^2+4x-5}{(1-x)(x-\sqrt{5-4x})}\\ &= \frac{(x+5)(x-1)}{(1-x)(x-\sqrt{5-4x})}\end{aligned}$$

At this point, we notice we have an $x-1$ term in the numerator and a $1-x$ term in the denominator. So we note that $1-x=-1(-1+x)=-(x-1)$, and therefore

$$\frac{(x+5)(x-1)}{(1-x)(x-\sqrt{5-4x})} = \frac{(x+5)(x-1)}{-(x-1)(x-\sqrt{5-4x})} = -\frac{(x+5)}{(x-\sqrt{5-4x})}.$$

Then

$$\lim_{x\to 1}\frac{x+\sqrt{5-4x}}{1-x} = \lim_{x\to 1}-\frac{(x+5)}{(x-\sqrt{5-4x})} = -\frac{((1)+5)}{((1)-\sqrt{5-4(1)})} = \frac{6}{1-1} = -\frac{6}{0}.$$

We notice now that we are still dividing by zero! Most problems of this sort resolve after just one conjugate multiplication (usually by the numerator). In this case, the issue we face now exists on the denominator. We will then multiply both the numerator and denominator by the conjugate of the denominator $x+\sqrt{5-4x}$ and note that

$$-\frac{(x+5)}{(x-\sqrt{5-4x})}\cdot\left(\frac{x+\sqrt{5-4x}}{x+\sqrt{5-4x}}\right) = -\frac{(x+5)(x+\sqrt{5-4x})}{x^2-(5-4x)} = -\frac{(x+5)(x+\sqrt{5-4x})}{x^2+4x-5}$$

We notice that the denominator still evaluates to zero at $x=1$, so we factor $x^2+4x-5=(x+5)(x-1)$ yet again and note that we now have

$$-\frac{(x+5)(x+\sqrt{5-4x})}{x^2+4x-5} = -\frac{(x+5)(x+\sqrt{5-4x})}{(x+5)(x-1)} = -\frac{(x+\sqrt{5-4x})}{(x-1)}$$

■

1.6 Continuity

So far, we've spoke only loosely about the concept of continuity. We will now put continuity on a more formal footing. There are three conditions for a function to be POINTWISE CONTINUOUS (or continuous at a point). We will give the definition, explain each of the conditions, and then graphically show the cases and define terms to help us discuss those cases before giving examples.

R Make note that if *any* of these conditions fail to hold, the function is discontinuous at the point given. In addition, a function being discontinuous at $x = a$ does *not* imply that it is discontinuous *everywhere*.

1.6.1 The Conditions of Continuity

Definition 1.6.1 A function $f(x)$ is continuous at $x = a$ if and only if:

1. $f(a)$ exists,
2. $\lim_{x \to a} f(x)$ exists, and
3. $\lim_{x \to a} f(x) = f(a)$.

Note 1.6.1 The first condition states that $f(a)$ must exist. That means that the function f must output a value when $x = a$ is input. So for $a = 4$ and $f(x) = \sqrt{x}$, $f(a)$ exists. But if $a = -2$ and $f(x) = \sqrt{x}$, $f(a)$ doesn't exist (assuming we are in $\mathbb{R}$, of course). The process for checking this condition is as simple as evaluating $f(x)$ at $x = a$ and determining if a valid output is obtained (as in, the function is defined to be a number there).

Note 1.6.2 The second condition states that $\lim_{x \to a} f(x)$ exists. This means that the left- and right-handed limits must equal. In notation,

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x).$$

The process for checking this condition is as simple as finding both the left- and right-handed limits and ensuring they are the same.

Note 1.6.3 The third and final condition states that

$$\lim_{x \to a} f(x) = f(a).$$

This means that the values obtained in the first two conditions must equal. The process for checking this condition is as simple as making sure the number obtained in the first condition and the two numbers obtained in the second condition are all three the same.

■ **Example 1.21** Tell if the piecewise-defined function below is continuous at $x = 0$. If it is not, tell which condition it failed.

$$f(x) = \begin{cases} x+1 & \text{if } x < 0 \\ x^2+1 & \text{if } x \geq 0 \end{cases}$$

First, we will note that we are testing only at the point $a = 0$. So we will check to see if the first condition is passed, which asks if $f(0)$ is defined:

$$f(0) = (0)^2 + 1 = 1.$$

R We plugged $a = 0$ into the bottom piece of the piecewise function because $0 \geq 0$. If the value had been less than zero, we would have used the top piece instead.

Now we will check the second condition. We must find both the left- and right-handed limits and ensure they are the same. We will check the left-handed limit first, noting that as we approach from the left side of zero, we are always less than zero and therefore will use the top piece of the piecewise function. We therefore obtain

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+1 = (0)+1 = 1.$$

Noting that as we approach from the right side of zero, we are always greater than zero, we will use the bottom piece of the piecewise function to find the right-handed limit. We therefore obtain

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+1 = (0)^2+1 = 1.$$

Since we obtained a limit value of $L = 1$ for both limits, the function passes the second condition. To see if the function passes the third condition, we simply note that we have obtained the same value (namely the value of one) for each of our calculations so for. Then

$$\lim_{x\to 0} f(x) = f(0)$$

and the function $f(x)$ is therefore continuous at $x = 0$. ■

Note 1.6.4 Your work on your paper for the previous example can be as simple as

1. $f(0) = 0^2 + 1 = 1$ ✓
2. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+1 = 0+1 = 1$ ✓
   $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2+1 = 1$ ✓
3. $f(x) = 1 + f(0) = 1$ ✓

The function $f(x)$ is therefore continuous at $x = 0$.

1.6.2 Graphical Intuition

We'll now take some time to review the terms for the various types of discontinuities. First, remember the informal definition of continuity, which is that a function is continuous at a point $x = a$ if and only if nothing strange happens there. If in the process of drawing the function, you simply moved your pencil through that point of the curve, it was continuous there. If you had to stop and pick your pencil up to draw a hole or an asymptote, it is discontinuous there. We will define a discontinuity formally now.

Definition 1.6.2 — Discontinuity. A discontinuity occurs in a function $f(x)$ at any point $x = a$ such that one of the three conditions for continuity fails.

There are several types of discontinuities, but we will focus on three. The JUMP DISCONTINUITY, the REMOVABLE DISCONTINUITY, and the INFINITE DISCONTINUITY. For each, we will define it, show a graphical example, and show a example problem where such a discontinuity occurs at that point.

Definition 1.6.3 — Jump Discontinuity. A jump discontinuity occurs in a function $f(x)$ at any point $x = a$ such that the second condition fails.

■ **Example 1.22** ■

Definition 1.6.4 — Removable Discontinuity. A removable discontinuity occurs in a function $f(x)$ at any point $x = a$ such that the second condition holds, but the first or third condition fails.

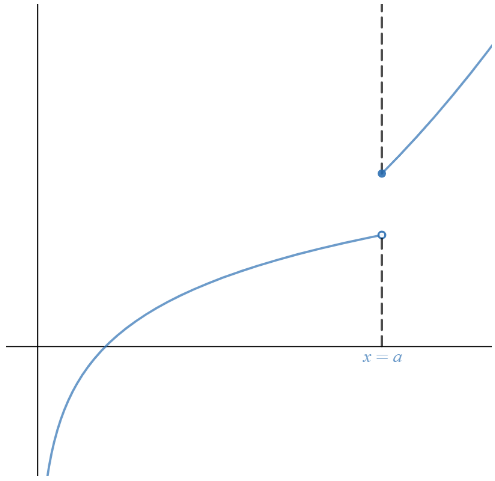

Figure 1.22: A jump discontinuity.

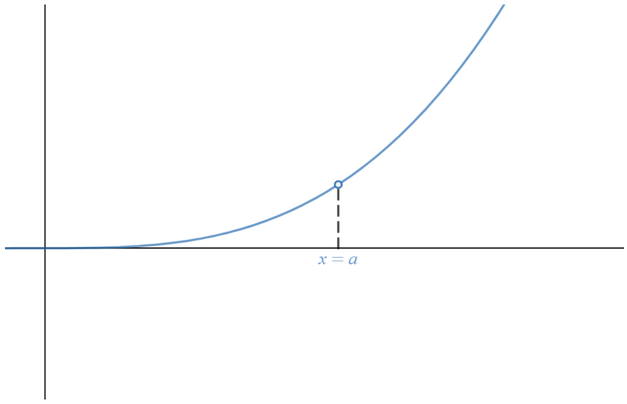

Figure 1.23: A removable discontinuity.

■ **Example 1.23** Tell if $f(x) = \frac{1-x}{x-1}$ is continuous at $x = 1$. If not, state which condition it fails.

First, we will note that $x = 1$ is our point a we wish to test. So we must check condition one to see if $f(1)$ exists.

$$f(1) = \frac{1-(1)}{(1)-1} = \frac{0}{0}.$$

Since $\frac{0}{0}$ is not defined, this condition is failed and we have no need to test further. This function is discontinuous at $x = 1$ and it failed condition one. ■

Exercise 1.40 Explain how the previous example was an example of a removable discontinuity and not an asymptote. ■

Definition 1.6.5 — Infinite Discontinuity. An infinite discontinuity occurs in a function $f(x)$ at any point $x = a$ such that there exists a vertical asymptote at that point.

■ **Example 1.24** Tell if $f(x) = \frac{1}{x-3}$ is continuous at $x = 3$. If not, state which condition it fails.

First, we check to see if $f(1)$ exists:

$$f(1) = \frac{1}{(3)-3} = \frac{1}{0}.$$

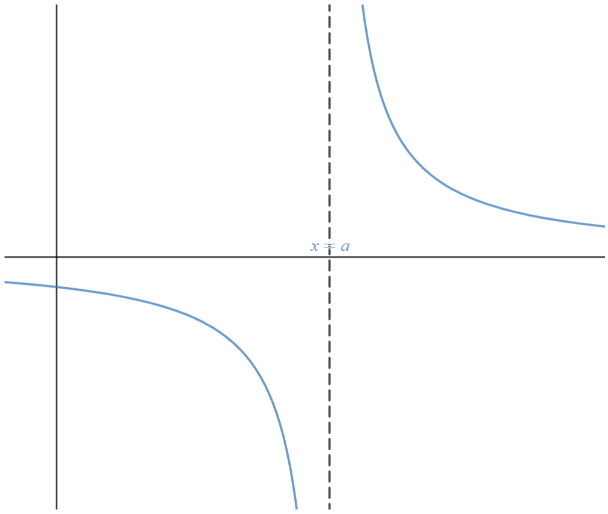

Figure 1.24: A vertical asymptote.

Since $\frac{1}{0}$ is undefined, this condition is failed and we have no need to test further. This function is discontinuous at $x = 3$ and it failed condition one. ■

Exercise 1.41 Explain how the previous example was an example of an asymptote and not a removable discontinuity. (Think about cancelling factors and look back to the previous exercise). ■

R It is possible for *only* the third condition to fail. This situation is always a removable discontinuity.

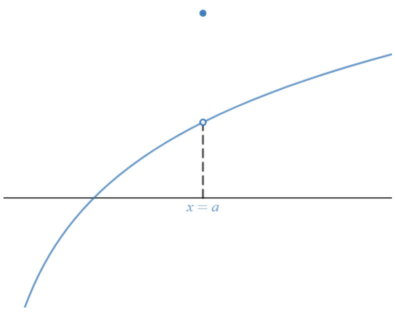

Figure 1.25: A place where only condition three fails.

Note 1.6.5 — Tricky Piecewise Functions. Consider the function

$$f(x) = \begin{cases} \sin(\tan(x)) & x < 0 \\ e^{2\pi \tan(x+i)} & x > 0. \end{cases}$$

If we were asked if the function is continuous at $x = 0$, it would fail condition one. Notice closely how neither piece of the piecewise-defined function holds at $x = 0$. If there is not a piece of the function to input your given value, it is not defined there. Graphically, this would be a gap in the graph. Because of this, we never even have to worry about how complex the bottom piece of the function is!

Exercise 1.42 Let $f(x)$ be a function such that $\lim_{x\to 1^-} f(x) = 2$, $\lim_{x\to 1^+} f(x) = 5$, and $f(1) = 5$. Is $f(x)$ continuous at the point $x = 1$? Why or why not? ■

Exercise 1.43 Let $f(x)$ be a function such that $\lim_{x\to -5^-} f(x) = 3$, $\lim_{x\to -5^+} f(x) = 3$, and $f(-5) = 3$. Is $f(x)$ continuous at the point $x = -5$? Why or why not? ■

Exercise 1.44 Let $f(x)$ be a function such that $\lim_{x\to 0^-} f(x) = -1$, $\lim_{x\to 0^+} f(x) = -1$, and $f(0) = 4$. Is $f(x)$ continuous at the point $x = 0$? Why or why not? ■

Exercise 1.45 Let $f(x)$ be a function such that $\lim_{x\to 7} f(x)$ does not exist, but $f(7) = 0$. Is $f(x)$ continuous at the point $x = 7$? Why or why not? ■

Exercise 1.46 Is the function $f(x)$ continuous at the point $x = 2$?

$$f(x) = \begin{cases} 18 - x^2, & x < 2 \\ 16 - x, & x \geq 2 \end{cases}$$

■

Exercise 1.47 Is the function $f(x)$ continuous at the point $x = 0$?

$$f(x) = \begin{cases} \sin(x) + 1, & x > 0 \\ 1 - x, & x < 0 \end{cases}$$

■

Exercise 1.48 Find the value k that makes the function continuous.

$$f(x) = \begin{cases} 10x - 139, & x > 7 \\ kx + 8, & x \leq 7 \end{cases}$$

■

1.6.3 Continuity of Linear Functions

Remember earlier in the text when we said we'd show later that linear functions are continuous everywhere in their domain? No? We did, I promise. Regardless, we'll show that now. (In fact, it's possible you've already showed this!) We will show $f(x) = mx + b$ is *everywhere* continuous, meaning it is continuous at all points in its domain.

Note 1.6.6 — The Proof Strategy. Using the definition of continuity, we must show that
1. $f(a)$ exists,
2. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, and
3. $\lim_{x\to a} f(x) = f(a)$.

Proof. Let $f(x) = mx + b$ with $a, b, m \in \mathbb{R}$.

First, we will show that $f(a)$ exists. Since the domain of f is $\mathbb{R}$, it is obvious that $a \in \mathbb{R}$ and therefore exists.

Next, we will show that

$$\lim_{x\to a^-} f(x) = ma + b = \lim_{x\to a^+} f(x).$$

Using the limit laws and recalling that $b, m \in \mathbb{R}$, we observe that the left-handed limit evaluates as

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (mx + b) = \left(\lim_{x\to a^-} m\right) \cdot \left(\lim_{x\to a^-} x\right) + \left(\lim_{x\to a^-} b\right) = ma + b.$$

Similarly, we can see that the process is much the same for the right-handed limit.

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (mx + b) = \left(\lim_{x\to a^+} m\right) \cdot \left(\lim_{x\to a^+} x\right) + \left(\lim_{x\to a^+} b\right) = ma + b.$$

Then we have shown the second condition holds. All that remains now is to show that

$$\lim_{x\to a} f(x) = f(a).$$

For this, we will need the rigor of an ε-δ proof, which means we need to find our δ in terms of an ε.

$$\begin{aligned}
|x - a| &< \delta \\
|x - a| &< \varepsilon/|m| \\
|m||x - a| &< \varepsilon \\
|mx - ma| &< \varepsilon \\
|mx + b - ma - b| &< \varepsilon \\
|(mx + b) - (ma - b)| &< \varepsilon \\
|f(x) - L| &< \varepsilon
\end{aligned}$$

So we need to choose $\delta = \frac{\varepsilon}{|m|}$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{|m|}$ such that $|x - a| < \delta$. Then

$$|f(x) - L| = |(mx + b) - (ma + b)| = |mx - ma| = |m||x - a| < |m|\delta = |m|\frac{\varepsilon}{|m|} = \varepsilon.$$

Therefore

$$\lim_{x\to a} mx + b = ma + b.$$

Then the three conditions are met and the function $f(x) = mx + b$ is continuous for all $a \in \mathbb{R}$. ■

Note 1.6.7 In fact, we can say the following are true:

1. All polynomial functions are continuous everywhere.
2. All rational functions are continuous on their domain.
3. The absolute value function is continuous everywhere.
4. The function $f(x) = \sqrt[n]{x}$ is continuous on its domain.
5. All six trigonometric functions are continuous over their respective domains.
6. If two functions f and g are continuous, then their arithmetic combinations are also continuous on their domains (that is, adding, subtracting, multiplying, dividing, raising to powers, or taking roots does not change that their result is also continuous).

$$\mathrm{I2} * \frac{\frac{\mathrm{I3}}{\mathrm{I4}}}{1} - \left(1 + \frac{\mathrm{I3}}{\mathrm{I4}}\right)^{(-\mathrm{I4} * \mathrm{I5})}$$

II

Derivatives

2. The Derivative

2.1 Development of the Derivative

War. War never changes. Everything else, however, does, and for that, we have Calculus. Many books on the subject define Calculus as 'the study of changes,' and they're not terribly off in their definition! The primary driving force behind this first course in Calculus is the derivative, which we will define shortly. Before that, let us talk about a secant line.

Definition 2.1.1 — The Secant Line. A secant line is a line that touches the graph of a function at two or more points.

In case we are in need of an illustration, observe the following:

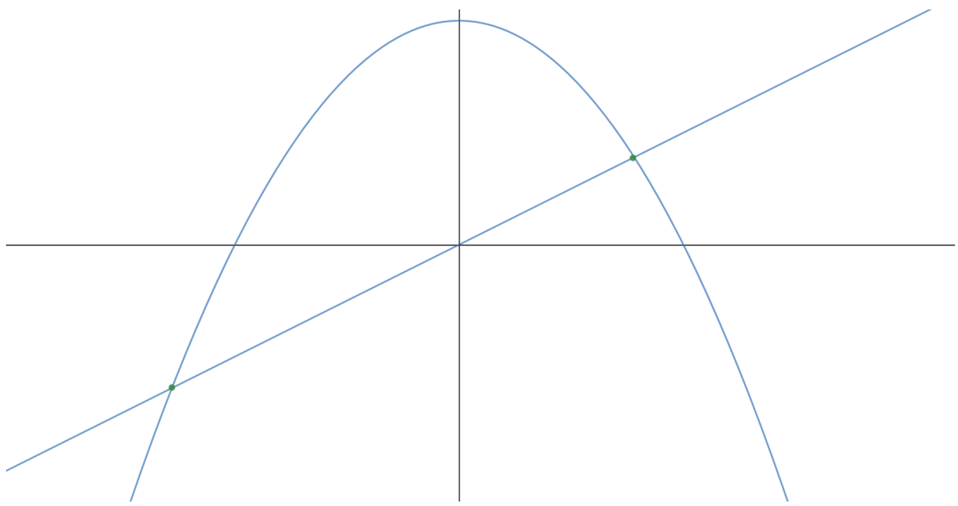

Figure 2.1: A secant line through a parabola.

Definition 2.1.2 — Average Rate of Change. We can find the *average rate of change* between the two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ by:

$$m_{avg} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

You may recall this particular formula (and why we use m for the variable) from a basic Algebra course. The average rate of change is, of course, just the mathematical name for slope, a concept you're already familiar with. That is, it's just the ratio of a function's change in y (or *rise*) to its change in x (or *run*). It's just in this notation instead of saying y_1 and y_2, we say $f(x_1)$ and $f(x_2)$, respectively. This works, since $f(x_1)$ obviously gives the height of the function f at point x_1.

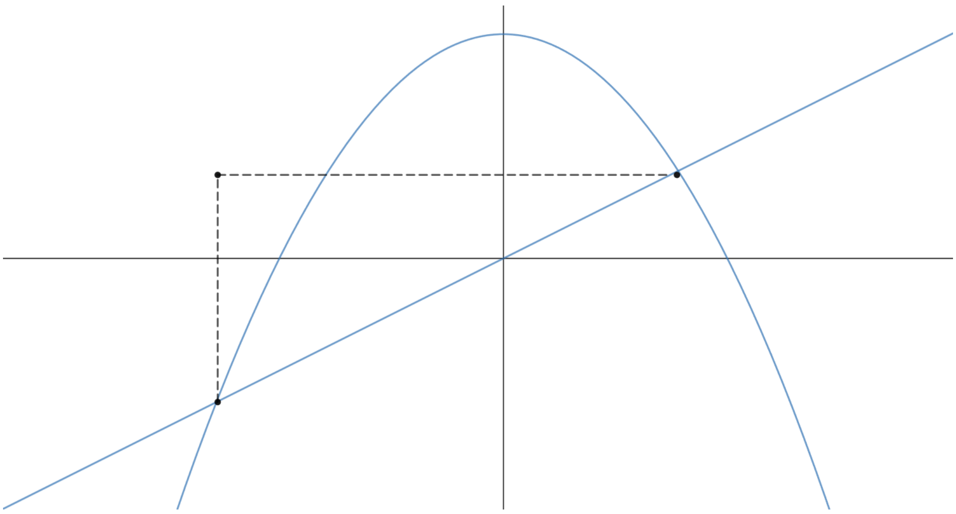

Figure 2.2: A secant line's changes in x and y illustrated.

Note 2.1.1 The Greek capital letter delta (Δ) is used to denote 'change in'. So Δx is read 'delta x,' and means the change in x between two points. Namely $\Delta x = x_2 - x_1$. The same goes for Δy.
In this way, with $y_2 = f(x_2)$ and $y_1 = f(x_1)$, we can say that

$$m_{avg} = \frac{\Delta y}{\Delta x}.$$

■ **Example 2.1** Find the average rate of change between the points $(7,1)$ and $(3,-1)$
We know from the problem that $(x_1, y_1) = (7,1)$. So $x_1 = 7$ and $f(x_1) = y_1 = 1$. By the same token, $x_2 = 3$ and $f(x_2) = y_2 = -1$. Then using the formula,

$$m_{avg} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{(-1)-(1)}{(3)-(7)} = \frac{-2}{-4} = \frac{1}{2}.$$

■

Change is happening in the world all around us constantly over a period of time. Over extremely long periods of time, planets are formed and life evolves. Over shorter, but still long, periods, we've seen the entirety of human civilization change from nothing to what it is today. On much shorter periods, human lives are lived, changing constantly. Shorter still are the day-to-day and minute-to-minute changes of physical and natural processes, and at the lowest currently-theorized level, strings of probability vibrate and change and give shape to everything.

That last bit is literally the basis for String Theory, a form of theoretical physics that is well beyond the scope of this course, but every form of science at its most basic is interested in the rate of change *in an instant*. That is, not over a period of finite time, but instead at a particular moment in time, as if freeze-framed or paused on a video.

If you think of a video of a car moving quickly along a racetrack and then pause it, how fast was the car going at that moment according to the picture? It's a paused picture of a car, right? So it's not moving, even though it may be blurry from the movement. Turns out we can get this information using the idea of a tangent line, which we'll define formally shortly.

Keeping in mind that mathematically we need two points $P_1 = (x_1, f(x_1))$ and $P_2 = (x_2, f(x_2))$, we're going to fix one point (P_1) in place and think about moving the other point (P_2) towards P_1.

To illustrate this, consider the following graph, with P_1 on the left and P_2 on the right:

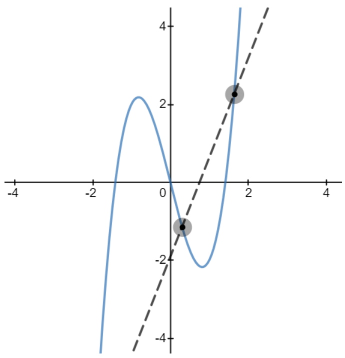

Figure 2.3: A secant line's changes in x and y illustrated.

Now, remember that we are leaving the left-most point, P_1, in place, and we are moving P_2 along the graph of f towards P_1.

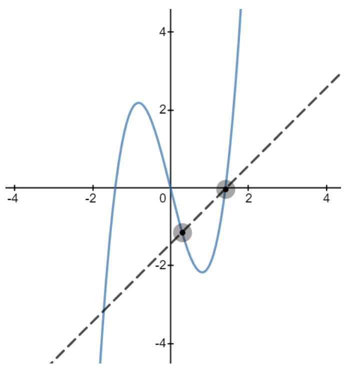

Figure 2.4: The points grow closer.

Continuing the process, we see the points getting even closer.

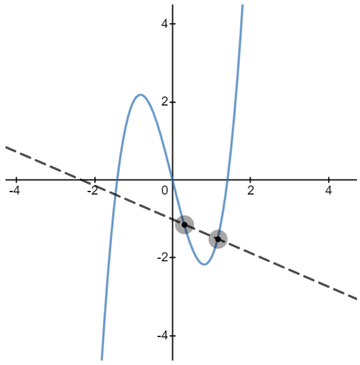

Figure 2.5: The points grow closer still...

Continuing the process, we see the points getting even closer still.

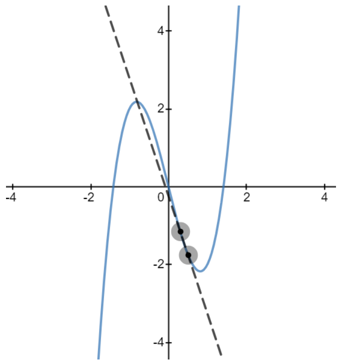

Figure 2.6: A secant line's changes in x and y illustrated.

Until eventually, those two points are so close to each other (in the limit) that they are the same point, as in figure 2.7 below.

Definition 2.1.3 — The Tangent Line. A *tangent line* touches a function at a single point and has the same slope as the function at that point.

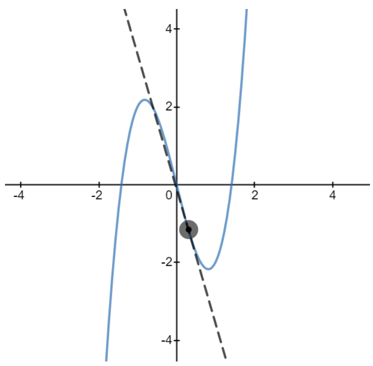

Figure 2.7: The points are basically the same.

2.1.1 Introduction to the Derivative

So taking this idea with the secant line ‘becoming’ a tangent line through limits and applying it to our rate of change formula, we let x_2 approach x_1 in a limit, and we have that

Definition 2.1.4 Assuming the limit exists, the derivative $f'(x)$ of the function $f(x)$ is given by

$$f'(x) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

That is read “f prime of x equals the limit as x_2 approaches x_1 of the quantity $f(x_2) - f(x_1)$ over the quantity $x_2 - x_1$.” We use the $'$ mark to indicate that the function bearing that mark is the derivative. This notation was used by Joseph-Louis Lagrange, and is one of three different notations that were all developed independently of one another.

This definition isn’t the most fun to work with, so let’s make a small change to the notation. If we let h be the distance between x_2 and x_1, then we have that $h = x_2 - x_1$ by definition, since $x_2 > x_1$ by convention. Then we notice that as $x_2 \to x_1$, the distance between them goes to zero. That is $h \to 0$. Solving $h = x_2 - x_1$ for x_2 gives us that $x_2 = h + x_1$, so that we have

$$f'(x) = \lim_{h \to 0} \frac{f(x_2 + h) - f(x_2)}{h}.$$

Since there’s only one x in that formula, the subscript is no longer needed. Thus, we arrive at our main limit definition of the derivative.

Definition 2.1.5 — The Limit Definition of the Derivative. Assuming the limit exists, the derivative $f'(x)$ of the function $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Easy enough to read, but let’s take a look at an example to see this in practice. Note that what we’re finding is the derivative, which is the slope of the line tangent to the curve given by some function f at a point x in its domain.

■ **Example 2.2** Find the derivative of $f(x) = 2x + 1$ at the point $x = 3$.

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{1}{h}[f(x+h) - f(x)] \\
&= \lim_{h \to 0} \frac{1}{h}[(2(x+h) + 1) - (2x + 1)] \\
&= \lim_{h \to 0} \frac{1}{h}[\cancel{2x} + 2h + \cancel{1} - \cancel{2x} - \cancel{1}] \\
&= \lim_{h \to 0} \frac{1}{h}[2h] \\
&= \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}} \\
&= \lim_{h \to 0} 2 \\
&= 2
\end{aligned}$$

So we wrote the definition out, noted that dividing by h is the same thing as multiplying the whole numerator by $\frac{1}{h}$, then we plugged in the actual values. So $f(x+h) = 2(x+h) + 1$, and

$f(x) = 2x+1$. Then we were careful with the distribution of the negative signs, cancelled like terms, canceled the h terms, then took the limit.

It's important to take a moment here and think about what we just found. While it's trivial in the end, we found that the slope of the function given by $f(x) = 2x+1$ at $x = 3$ is 2. This is trivial, of course, because the slope of that line is *always* 2, but it does show that the mathematics works, and is a decent first example. ■

Exercise 2.1 Find the slope of the function $f(x) = 4x-2$ at $x = 6$. ■

Now let's try this in general with a more difficult function like a quadratic. When we say 'in general,' we mean not at a specific point.

■ **Example 2.3** Find $f'(x)$ if $f(x) = x^2+2x-3$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{1}{h}[((x+h)^2+2(x+h)-3)-(x^2+2x-3)] \\
&= \lim_{h\to 0}\frac{1}{h}[\cancel{x^2}+2xh+h^2+\cancel{2x}+2h-\cancel{3}-\cancel{x^2}-\cancel{2x}+\cancel{3}] \\
&= \lim_{h\to 0}\frac{1}{h}\left[2xh+h^2+2h\right] \\
&= \lim_{h\to 0}\left[\frac{2x\cancel{h}}{\cancel{h}}+\frac{h^2}{h}+\frac{2\cancel{h}}{\cancel{h}}\right] \\
&= \lim_{h\to 0}[2x+h+2] \\
&= 2x+(0)+2 \\
&= 2x+2.
\end{aligned}
$$

In this example, as in the last, we plugged in $f(x+h)$ and $f(x)$ to our limit definition of the derivative. $f(x)$ is given to us, but note that

$$f(x+h) = (x+h)^2+2(x+h)-3.$$

That is, it is simply $f(x) = x^2+2x-3$ with $(x+h)$ instead of x terms. Be very careful to plug in your $f(x)$ with grouping symbols around it, otherwise your minus from the definition won't get distributed correctly!

Once everything is plugged in, these problems always boil down to canceling all the terms that don't have an h in them, then distributing through that division by h from the formula and evaluating the limit.

Notice in the end we get that $f'(x) = 2x+2$. That derivative is a formula that will give us the slope of the line tangent to $f(x)$ *for any x we plug in.* This is tremendously powerful and useful. ■

Exercise 2.2 Find the derivative $f'(x)$ given $f(x) = x^2-5x+4$. ■

Let's do one more example, and let us go ahead and state a careful version of the limit definition that we've already used twice:

Definition 2.1.6 — A Careful Limit Definition of the Derivative. Assuming the limit exists, the derivative $f'(x)$ of the function $f(x)$ is given by

$$f'(x) = \lim_{h\to 0}\left(\frac{1}{h}\left[(f(x+h))-(f(x))\right]\right).$$

Notice how we wrote the division by h as a multiplication by $\frac{1}{h}$, along with including grouping parentheses around the two functions. Plugging in to this form will minimize bookkeeping errors (minor mistakes).

One final example should do us for this concept.

■ **Example 2.4** Find $f'(x)$ given $f(x) = x^3$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\left(\frac{1}{h}\left[(x+h)^3 - (x)^3\right]\right) \\
&= \lim_{h\to 0}\frac{1}{h}[\cancel{x^3} + 2x^2h + xh^2 + x^2h + 2xh^2 + h^3 - \cancel{x^3}] \\
&= \lim_{h\to 0}\left[\frac{2x^2\cancel{h}}{\cancel{h}} + \frac{x\cancel{h^2}}{\cancel{h}} + \frac{x^2\cancel{h}}{\cancel{h}} + \frac{2x\cancel{h^2}}{\cancel{h}} + \frac{\cancel{h^3}}{\cancel{h}}\right] \\
&= \lim_{h\to 0}[2x^2 + xh + x^2 + 2xh + h^2] \\
&= \lim_{h\to 0}[3x^2 + 3xh + h^2] \\
&= [3x^2 + 3x(0) + (0)^2] \\
&= 3x^2.
\end{aligned}$$

Using the careful definition, we plugged in $f(x+h) = (x+h)^3$ and $f(x) = x^3$ and then expanded the $(x+h)^3$ term using Algebra by writing $(x+h)^3 = (x+h)(x+h)(x+h)$ and following the binomial expansion rules. From there, we canceled the non-h terms as usual, distributed the $\frac{1}{h}$ term, canceled the h terms we could, then took the limit to finish. ■

Note 2.1.2 — The Process of Taking a Derivative Using the Limit Definition. There is a process that can be followed with these, which is a pattern we will pick up on as we do more of them. To wit:

1. Find $f(x+h)$ by plugging in $x+h$ to f and doing whatever math is needed to simplify fully. This is usually just binomial expansion (FOIL method).
2. Cancel all non-h terms. If any terms without an h survive, something is wrong, check your math.
3. Distribute the $\frac{1}{h}$ term to each remaining term.
4. Cancel an h term everywhere it is possible.
5. Evaluate the remaining limits, remembering that $\lim_{h\to 0} h = 0$.
6. Simplify the remaining arithmetic.
7. The function given is the derivative.
8. (Optional) If a value is given to evaluate at, plug that number into the function we obtained from this process.

Exercise 2.3 Find $f'(x)$ given $f(x) = x^3 - x^2 + 1$. ■

2.1.2 Finding the Equation of a Tangent Line

We saw in earlier figures that a derivative is just the limit of the slope of a secant line as one point approaches the second. We will now use the fact that the derivative is just the slope of a line tangent to find the entire equation of a tangent line. This is simple enough if we recall from a basic algebra class.

Definition 2.1.7 — The Point-Slope Form of a Line. The point-slope form of a line through the point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1).$$

For our needs with a function $f(x)$, we have a point $(a, f(a))$, and we know that the slope is just the derivative of f at a, denoted $f'(a)$. Then $m = f'(a)$ and our formula becomes

Definition 2.1.8 — The Form of a Tangent Line. The point-slope form of a line tangent to a differentiable function $f(x)$ through the point $(a, f(a))$ with slope $f'(a)$ is

$$f(x) - f(a) = f'(a)(x - a),$$

or just

$$f(x) = f'(a)(x - a) + f(a).$$

Note 2.1.3 Recall that

$$f'(a) = \lim_{h \to 0} \left(\frac{1}{h} [(f(a+h)) - (f(a))] \right).$$

It is literally just Definition 2.1.6 with $x = a$.

■ **Example 2.5** Find the equation of the tangent line to $f(x) = x^2$ at $a = 4$. In order to do this, we need to know the point $(a, f(a))$ and the derivative at a (which is the slope at that point) $f'(a)$. First, we know $a = 4$, so we have that $(a, f(a)) = (4, f(4)) = (4, (4)^2) = (4, 16)$. Then $f(a) = f(4) = 16$. That means our final answer will look something like

$$f(x) = f'(4)(x - 4) + 16.$$

This is the second form of Definition 2.1.8 with $f(x) = x^2$ and $a = 4$. Clearly all that is left is to find $f'(4)$. First we find $f'(a)$, then we plug in $a = 4$ and we're done! Finding $f'(a)$ is a bit tedious, but we develop rules later on that help. Let's get started.

$$\begin{aligned}
f'(a) &= \lim_{h \to 0} \left(\frac{1}{h} \left[(a+h)^2 - (a)^2 \right] \right) \\
&= \lim_{h \to 0} \left(\frac{1}{h} \left[(a^2 + 2ah + h^2 - a^2 \right] \right) \\
&= \lim_{h \to 0} \left(\frac{1}{h} \left[2ah + h^2 \right] \right) \\
&= \lim_{h \to 0} \left[\frac{2ah}{h} + \frac{h^2}{h} \right] \\
&= \lim_{h \to 0} (2a + h) \\
&= 2a + (0) \\
&= 2a.
\end{aligned}$$

Then $f'(4) = 2(4) = 8$ and our final equation is

$$f(x) = 8(x - 4) + 16 = 8x - 32 + 16 = 8x - 16,$$

and we would write our answer as simply

$$f(x) = 8x - 16.$$

■

Note 2.1.4 Note how the answer to the question was an equation of $f(x)$, not our equation $f'(a)$. Graph $f(x) = x^2$ and $g(x) = 8x - 16$ and look at the point $(8, 16)$. The question was asking for the equation that gives the line tangent at the point, which is what this note calls $g(x)$. If $f'(a) = 2a$ was given as an answer, that would be giving the equation of the slope of the line at a, which is obviously not what is being asked for. Be careful with this.

2.1.3 Linear Approximation of a Function

While we're on the topic, we should take the time to discuss a concept in mathematics that is better shown through an exercise.

Exercise 2.4 Graph $y = x^2 + 2x + 1$ on a graphing utility. While you have it pulled up, choose any section of the curve and zoom in very close at the point? What shape is the curve? Try zooming in at another point. What shape is the curve there? ■

If you said the shape is just a line, then you understand the concept of linear approximation! In fact, if we recall Definition 1.3.1 which defined the concept of a neighborhood, we could say the following:

Definition 2.1.9 — The Linearization of a Function at a Point. Given a function $f(x)$ that is differentiable at the point $(a, f(a))$, the linear approximation or linearization of f at a is given by

$$L(x) = f'(a)(x - a) + f(a).$$

2.2 Basic Derivative Rules

Like any topic in mathematics, there are a few rules or laws that come from the definition that are going to help us. The first is the Addition Rule (which covers subtraction because subtraction is just addition of a negative number).

2.2.1 The Addition Rule

Theorem 2.2.1 — Addition Rule, Lagrange Notation. Let f and g be differentiable functions. Then,

$$[f(x) \pm g(x)]' = f'(x) \pm g'(x)$$

For your edification, we will prove this. It should help explain how this law isn't just passed down from on high, never to be questioned. It is rather a direct result of the mathematics we already know.

Proof. For a function $j(x)$, its derivative is

$$j'(x) = \lim_{h \to 0} \frac{j(x+h) - j(x)}{h}$$

by definition. Note that we use $j(x)$ instead of $f(x)$ here only because we will be using f and g in the addition.

$$\begin{aligned}
[f(x) \pm g(x)]' &= \lim_{h\to 0} \frac{(f(x+h) \pm g(x+h)) - (f(x) \pm g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(x+h) \pm g(x+h) - f(x) \mp g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{(f(x+h) - f(x)) \pm (g(x+h) - g(x))}{h} \right] \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \pm \frac{g(x+h) - g(x)}{h} \right] \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(x) \pm g'(x)
\end{aligned}$$

■

We first used the definition of the derivative with $j(x) = f(x) \pm g(x)$. In the second step, we distributed through the negative sign, noting that $-(f(x) \pm g(x)) = -f(x) \mp g(x)$. That is, a negative being distributed simply changes the signs. In the next step, we gathered our like terms. In the next step, we wrote those terms separately over their common denominator of h. In the penultimate step, we separated the limit using a limit law, and then we noticed that each of the separate parts were their own definitions of the derivative in the last step.

Alternatively, we may state the last law in the notation preferred by Leibniz,

Theorem 2.2.2 — Addition Rule, Leibniz Notation. Let f and g be differentiable functions. Then,

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x) \pm g(x)] = \frac{\mathrm{d}f}{\mathrm{d}x} \pm \frac{\mathrm{d}g}{\mathrm{d}x}.$$

2.2.2 The Constant Derivative Rule

The second of the three rules is perhaps the simplest.

Theorem 2.2.3 — Constant Derivative Rule. The derivative of a constant is zero.

Proof. If $c \in \mathbb{R}$ and $f(x) = c$, then note that $f(x+h) = c$ and

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(c) - (c)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

■

2.2.3 The Constant Multiple Rule

We will now define one more basic rule that says we can factor or 'pull out' a constant from a derivative, a fact that we will relentlessly exploit until the heat death of the universe, saving us countless hours.

Theorem 2.2.4 — Constant Multiple Rule, Lagrange Notation. If $c \in \mathbb{R}$ and f is a differentiable function, then

$$(c \cdot f(x))' = cf'(x).$$

Theorem 2.2.5 — Constant Multiple Rule, Leibniz Notation. If $c \in \mathbb{R}$ and f is a differentiable function, then

$$\frac{\mathrm{d}}{\mathrm{d}x}[c \cdot f(x)] = c \cdot \frac{\mathrm{d}f}{\mathrm{d}x}.$$

Proof. We first note that

$$c \cdot f'(x) = c \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} c \cdot \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h}.$$

Since we have that

$$\lim_{h \to 0} \frac{c \cdot f(x+h) - c \cdot f(x)}{h} = [c \cdot f(x)]',$$

we are done.

■

Insofar as simple rules go, those are the only two that clearly come from the definition with clear and concise proofs. In the subsequent sections, we will discuss more advanced and robust rules that will allow us to handle taking the derivative of any function.

2.3 The Power Rule

We begin our exploration of more advanced derivative rules with the Power Rule, the most basic of these. Traditionally, this rule is defined only for powers in the natural numbers at first, then extended using the Quotient Rule once it's learned. However, we will go ahead and define it for all real numbers.

Theorem 2.3.1 — The Power Rule, Lagrange Notation. For $n \in \mathbb{R}$ and $f(x) = x^n$,

$$f'(x) = nx^{n-1}.$$

Theorem 2.3.2 — The Power Rule, Leibniz Notation. For $n \in \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^n] = nx^{n-1}.$$

Proof. The proof of the Power Rule is left until the end of this chapter, as it uses both implicit and logarithmic differentiation. ■

The proof for the Power Rule requires use of the Binomial Theorem, and is well-documented elsewhere. To extend it to the real numbers as we have above, the Quotient Rule (or Product and Chain Rules together) is required. As such, we omit the proof here. We will, however, do a few examples.

■ **Example 2.6** Given $f(x) = x^4$, find $f'(x)$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} x^4 = 4x^{4-1} = 4x^3.$$

■

■ **Example 2.7** Find $f'(x)$ if $f(x) = 2x^{23}$.

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} 2x^{23} = 23 \cdot 2x^{23-1} = 46x^{22}.$$

■

Note 2.3.1 Note how we had to multiply the two coefficients together out front after brought down the 23. We could have instead factored out the 2, like so

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} 2x^{23} = 2 \cdot \frac{\mathrm{d}}{\mathrm{d}x} x^{23} = 2 \cdot \left(23x^{23-1}\right) = 46x^{22}.$$

Note 2.3.2 — Using the Power Rule. There are a few simple steps to using the Power Rule. Since it only works on monomial terms as defined (more on that next), we will always be given something of the form $f(x) = cx^n$. The steps for finding $f'(x)$ are as follows, in this order:

1. Bring down the power n as a multiplication out front.
2. Reduce the power n by one.
3. Multiply the n we brought down by the c that was given.

Theorem 2.3.3 — The Power Rule for Polynomials. Since a polynomial is a summation of two or more monomials, we may simply apply the Power Rule to each part, as allowed by the Addition Rule.

■ **Example 2.8** Find $f'(x)$ given $f(x) = 2x^4 + 3x^2 - x + 734$.

$$\begin{aligned} f'(x) &= \tfrac{\mathrm{d}}{\mathrm{d}x}[2x^4 + 3x^2 - x + 734] \\ &= \tfrac{\mathrm{d}}{\mathrm{d}x}[2x^4] + \tfrac{\mathrm{d}}{\mathrm{d}x}[3x^2] - \tfrac{\mathrm{d}}{\mathrm{d}x}[x] + \tfrac{\mathrm{d}}{\mathrm{d}x}[734] \\ &= (4 \cdot 2x^{4-1}) + (2 \cdot 3x^{2-1}) - (1 \cdot x^{1-1}) + (0) \\ &= 8x^3 + 6x - 1 \end{aligned}$$

■

■ **Example 2.9** Find the equation of the tangent line to (or linear approximation of) $f(x) = x^2$ at $a = 4$. You'll notice this was the same problem we did a while back, but now that we have the power rule, it's much quicker. We note that if $f(x) = x^2$, then $f(a) = a^2$ and $f\prime(a) = 2a^{2-1} = 2a^1 = 2a$ and using the second part of Definition 2.1.8 we have that our equation must be

$$f(x) = f'(a)(x - a) + f(a),$$

and since $a = 4$, we have $f(4) = 4^2 = 16$, so

$$f(x) = (2a)(x - a) + f(a) = (2(4))(x - 4) + 16 = 8(x - 4) + 16 = 8x - 32 + 16 = 8x - 16,$$

as before. ■

2.4 Function-Derivative Relationships

So far, we have defined the derivative of a function at a point $x = a$ to be the slope of the line tangent to the function at that point. However, when viewing the graph of both a function and its derivative on the same plot, a few surprising facts emerge.

Since we're going to do this, we should define a function and its derivative. So we will let $f(x) = x^3 - x$. Then $f'(x) = 3x^2 - 1$. The graph of Figure 2.8 below shows both of these functions on the same plot. We see $f(x)$ plotted as the solid black line, and $f'(x)$ as the dashed function in red and blue.

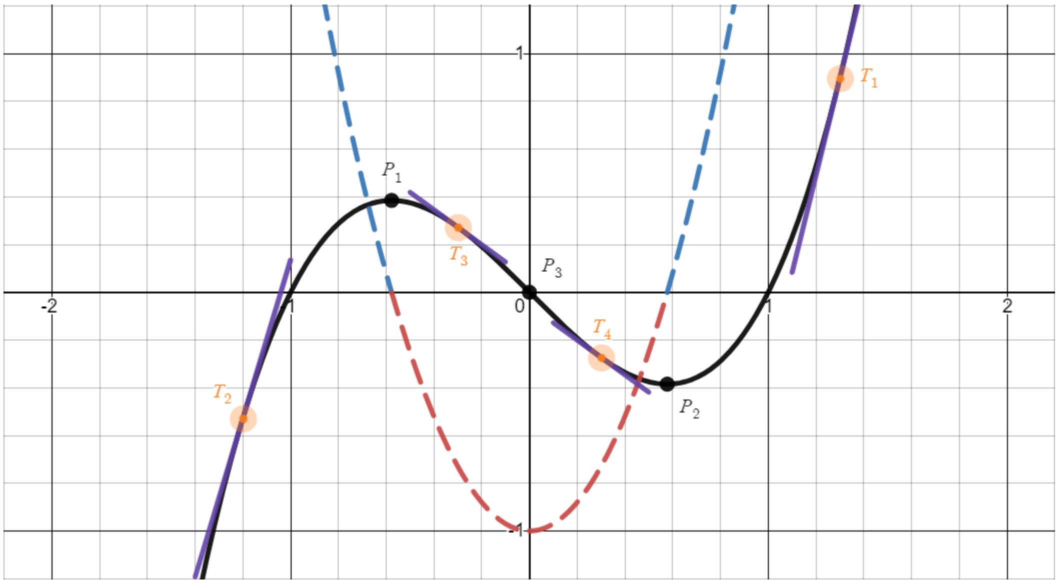

Figure 2.8: A graph of f and f'.

There is a lot to take in for this graph, but it is most important that we understand that the graph of f' (the parabola graphed by the dashed red-and-blue curve) tells us information about the slope of the tangent lines of f.

Look closely at the graph of f in black. There are three points labeled on it, P_1, P_2, and P_3. We know P_1 and P_2 are the local maximum and local minimum values of f, respectively. Moreover, we notice that the slope of the line tangent to f is zero at both of those points.

To the left of P_1 along the graph of f, we notice that the slopes of the tangent lines are all positive as we approach P_1 from the left. Moreover, we note that the slopes, while staying positive, are decreasing towards zero.

If we look at the graph of f' to the left of its leftmost x-intercept, we notice that it is always positive, but decreasing. This is the same behavior as the slopes of the tangent lines of f.

We notice between P_1 and P_2 the slopes of the tangent lines are negative. We also notice that the values of f' are all negative. However, at the point P_3 on the solid black graph of f, the tangent lines shift from getting more negative to getting more positive. We therefore notice that the graph of f' shifts from decreasing to increasing at the same point.

Exercise 2.5 Use the graph of $f'(x) = (x-1)^3$ to sketch a rough graph of $f(x)$. ■

2.5 Derivatives of Trigonometric Functions

As we know from a standard Precalculus course, all of the six trigonometric functions are based on the sine and cosine functions.

The proofs for $\tan x$, $\cot x$, $\sec x$, and $\csc x$ require the Quotient Rule, so their derivations are included in that section as extra examples. Most Calculus texts introduce these derivatives at a point later than this. We include them here so that there are more interesting problems that can be given to practice the Product and Quotient Rules, which we will learn soon.

Note 2.5.1 — Helpful Limits. The proofs for the derivatives of sine and cosine both make use of the fact that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

and

$$\lim_{x\to 0} \frac{\cos(x)-1}{x} = 0.$$

We now give the derivatives directly.

Definition 2.5.1 — The Derivative of Sine. If $f(x) = \sin x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x.$$

Proof. Let $f(x) = \sin x$. Then using the definition of the derivative gives us that

$$f'(x) = \lim_{h\to 0} \frac{1}{h}[\sin(x+h) - \sin(x).]$$

We recall that $\sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$, and use this to obtain that

$$f'(x) = \lim_{h\to 0} \frac{1}{h}[(\sin(x)\cos(h) + \cos(x)\sin(h)) - \sin(x).]$$

Grouping together the $\sin(x)\cos(h)$ and $-\sin(x)$ terms first and factoring out a $\sin(x)$ gives us that

$$f'(x) = \lim_{h\to 0} \frac{1}{h}[\sin(x)(\cos(h)-1) + \cos(x)\sin(h).]$$

Then using the Addition Law for limits and distributing in the $\frac{1}{h}$ terms gives us that

$$f'(x) = \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}\right) + \lim_{h\to 0}\left(\cos(x)\frac{\sin(h)}{h}\right).$$

We have grouped the division by h with the trigonometric terms that have to do with h as well, keeping in mind the limits we mentioned before. We will use the Product Law for limits on both terms to obtain that

$$f'(x) = \sin(x)\left(\lim_{h\to 0}\frac{\cos(h)-1}{h}\right) + \cos(x)\left(\lim_{h\to 0}\frac{\sin(h)}{h}\right).$$

Then we use the limits from Note 2.5.1 on the limits in parenthesis to obtain

$$f'(x) = \sin(x)(0) + \cos(x)(1) = \cos(x).$$

■

Definition 2.5.2 — The Derivative of Cosine. If $f(x) = \cos x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x.$$

Proof. The proof for the derivative of $\cos x$ uses a different trignometric formula, but is otherwise nearly identical to the proof for $\sin x$. It is therefore left as an exercise to the reader. ■

Exercise 2.6 Prove that $\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x$ using the limit definition of the derivative, together with Note 2.5.1 and the fact that

$$\cos(x+h) = \cos(x)\cos(h) - \sin(x)\sin(h).$$

■

Definition 2.5.3 — The Derivative of Tangent. If $f(x) = \tan x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x.$$

Definition 2.5.4 — The Derivative of Cotangent. If $f(x) = \cot x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x.$$

Definition 2.5.5 — The Derivative of Secant. If $f(x) = \sec x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x.$$

Definition 2.5.6 — The Derivative of Cosecant. If $f(x) = \csc x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x.$$

2.5.1 Table of Trigonometric Derivatives

Table 2.9 below may be of use if we find ourselves in need of looking up a derivative quickly.

2.6 The Product Rule

In a perfect world, the Product Rule would work like the Addition Rule. Let us take a look at why it does not work out to be so nice.

Assume we wanted to find the derivative of $f(x) = x^5$. By the Power Rule, that is simple enough:

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}x^5 = 5x^4.$$

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cdot \cot x$
$\sec x$	$\sec x \cdot \tan x$
$\cot x$	$-\csc^2 x$

Figure 2.9: A Table of Trigonometric Derivatives

Note now that $x^5 = x^3 \cdot x^2$. If the Product Rule worked like the Addition Rule, then the derivative of the products would be the product of the derivatives, like so:

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^3 \cdot x^2] = \frac{\mathrm{d}}{\mathrm{d}x}x^3 \cdot \frac{\mathrm{d}}{\mathrm{d}x}x^2 = 3x^2 \cdot 2x = 6x^3.$$

You may notice, however, that we obtained two different answers! Then the Product Rule must be more complicated. That is unfortunate, but true.

Theorem 2.6.1 — The Product Rule - Lagrange Notation. For differentiable functions f and g,

$$[f(x) \cdot g(x)]' = [f'(x)](g(x)) + (f(x))[g'(x)].$$

Theorem 2.6.2 — The Product Rule - Leibniz Notation. For differentiable functions f and g,

$$\frac{\mathrm{d}}{\mathrm{d}x}[f \cdot g] = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}.$$

Proof. As with the Power Rule, we leave the proof of the Product Rule until we have better tools with which to prove it. ■

So to apply this new rule to our example of $\frac{\mathrm{d}}{\mathrm{d}x}[x^3 \cdot x^2]$, we see that

■ **Example 2.10**

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}[x^3 \cdot x^2] &= [x^3]' \cdot (x^2) + (x^3) \cdot [x^2]' \\
&= 3x^2 \cdot x^2 + x^3 \cdot 2x \\
&= 3x^4 + 2x^4 \\
&= 5x^4.
\end{aligned}$$

■

Note 2.6.1 — The Gardner Notation for Product Rule. To borrow a method from the legendary Dr. Robert Gardner, the basic 'form' of a Product Rule is always going to look like

$$\frac{\mathrm{d}}{\mathrm{d}x}[()()] = []() + ()[].$$

We use square brackets to denote that a derivative was taken, and parenthesis to notate that the function was left as-is. It is a common mistake to assume that square brackets indicate that a derivative *should be* taken, but that is not the case! We still must use Lagrange or Leibniz's notation to indicate such a thing. Nevertheless, we will use square brackets to keep track of when derivatives are taken throughout the rest of the text, but that will not preclude us from using them in other situations. As always, context is king.

■ **Example 2.11** This example uses the Product Rule to find a derivative.

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}\left[(x+2)^2\right] &= \tfrac{\mathrm{d}}{\mathrm{d}x}[(x+2)(x+2)] \\
&= [x+2]'\cdot(x+2)+(x+2)\cdot[x+2]' \\
&= [1+0](x+2)+(x+2)[1+0] \\
&= (x+2)+(x+2) \\
&= 2x+4.
\end{aligned}$$

■

■ **Example 2.12** This example uses the Power Rule to find the derivative after expanding the binomial.

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}[(x+2)^2] &= \tfrac{\mathrm{d}}{\mathrm{d}x}[x^2+4x+4] \\
&= \tfrac{\mathrm{d}}{\mathrm{d}x}x^2+\tfrac{\mathrm{d}}{\mathrm{d}x}4x+\tfrac{\mathrm{d}}{\mathrm{d}x}4 \\
&= 2x+4+0 \\
&= 2x+4.
\end{aligned}$$

■

■ **Example 2.13** Find the derivative of $h(x)=2\sin x\cos x$.

To find $h'(x)$ here, we simply group our terms into two groups as $h(x)=(2\sin x)(\cos x)$. Then by the Product Rule, our derivative must look like

$$h'(x)=\frac{\mathrm{d}}{\mathrm{d}x}[2\sin x](\cos x)+(2\sin x)\frac{\mathrm{d}}{\mathrm{d}x}[\cos x].$$

Since we know $\frac{\mathrm{d}}{\mathrm{d}x}[2\sin x]=2\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]$, we know $\frac{\mathrm{d}}{\mathrm{d}x}[2\sin x]=2\cos x$, since $\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]=\cos x$. In the same way, we know that $\frac{\mathrm{d}}{\mathrm{d}x}[\cos x]=-\sin x$, so plugging this in, we have that

$$h'(x)=\frac{\mathrm{d}}{\mathrm{d}x}[2\sin x](\cos x)+(2\sin x)\frac{\mathrm{d}}{\mathrm{d}x}[\cos x]=[2\cos x](\cos x)+(2\sin x)[-\sin x].$$

Then gathering like terms we have that

$$h'(x)=2\cos^2 x-2\sin^2 x.$$

■

Exercise 2.7 Given $f(x)=\tan x\cdot\sin x$, find $f'(x)$. ■

Exercise 2.8 Given $f(x)=x^2\sec x$, find $f'(x)$. ■

Exercise 2.9 Given $f(x)=\sqrt{x}\cos x$, find $f'(x)$. ■

Exercise 2.10 Given $j(x) = f(x)g(x)h(x)$, find $j'(x)$. ■

2.7 The Quotient Rule

Definition 2.7.1 — The Quotient Rule - Lagrange Notation. Let f and g be differentiable functions such that $g \neq 0$. Then

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

(R) A useful (if mathematically juvenile) mnemonic to remember the Quotient Rule is "low d high minus high d low over the square of what's below." That is, g is 'low' and f is 'high.'

Definition 2.7.2 — The Quotient Rule - Leibniz Notation. Let f and g be differentiable functions such that $g \neq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f}{g}\right] = \frac{g \cdot \frac{\mathrm{d}f}{\mathrm{d}x} - f \cdot \frac{\mathrm{d}g}{\mathrm{d}x}}{g^2}.$$

Proof. We will leave the proof of the Quotient Rule until the end of the chapter as well, though it only needs the Product and Chain Rules to prove it. ■

■ **Example 2.14** We will now take the derivative of the rational function $\frac{x+1}{x+2}$. Note that $f(x) = x+1$ and $g(x) = x+2$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x+1}{x+2}\right] &= \frac{(x+2)[x+1]' - (x+1)[x+2]'}{(x+2)^2} \\
&= \frac{(x+2)[1] - (x+1)[1]}{(x+2)^2} \\
&= \frac{x+2-(x+1)}{(x+2)^2} \\
&= \frac{x+2-x-1}{(x+2)^2} \\
&= \frac{1}{(x+2)^2}.
\end{aligned}$$

We set up the problem in the first line. In the second line, we noted the derivatives of the linear functions $x+2$ and $x+1$ were both one. The rest of the problem was distributing the negative across and gathering like terms. As always with these problems, the hardest part is the setup!

■

Exercise 2.11 Use the Quotient Rule to find the derivative of $h(x) = \dfrac{3x^2}{1-4x}$. ■

Exercise 2.12 Use the Quotient Rule to find the derivative of $h(x) = \dfrac{\sin x}{\cos x}$. ■

2.7.1 Proofs of the Nonbasic Trigonometric Derivatives

As promised earlier, we now state and prove each of the derivatives for $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

Definition 2.7.3 — The Derivative of Tangent. If $f(x) = \tan x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x.$$

Proof. To find the derivative of the tangent function, we first note simply that $\tan x = \frac{\sin x}{\cos x}$ by definition. We then apply the Quotient Rule, making use of the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] \\
&= \frac{(\cos x)[\sin x]' - (\sin x)[\cos x]'}{(\cos x)^2} \\
&= \frac{\cos x \cdot \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x.
\end{aligned}$$

■

Definition 2.7.4 — The Derivative of Cotangent. If $f(x) = \cot x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x.$$

Proof. As with the tangent function, we first note simply that $\cot x = \frac{\cos x}{\sin x}$ by definition. We then apply the Quotient Rule, making use of the Pythagorean Identity $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\cot x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos x}{\sin x}\right] \\
&= \frac{(\sin x)[\cos x]' - (\cos x)[\sin x]'}{(\sin x)^2} \\
&= \frac{\sin x \cdot -\sin x - \cos x \cdot \cos x}{\sin^2 x} \\
&= \frac{-\sin^2 x - \cos^2 x}{\sin^2} \\
&= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} \\
&= \frac{-1}{\sin^2 x} \\
&= -\csc^2 x.
\end{aligned}$$

■

Definition 2.7.5 — The Derivative of Secant. If $f(x) = \sec x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x \tan x$$

Proof. To find the derivative of the secant function, we first note simply that $\sec x = \frac{1}{\cos x}$ by definition. We then apply the Quotient Rule with $f(x) = 1$ and $g(x) = \cos x$, making use of simple trigonometric quotient and reciprocal identities.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sec x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\cos x}\right] \\
&= \frac{(\cos x)[1]' - (1)[\cos x]'}{\cos^2 x} \\
&= \frac{0 - (-\sin x)}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} \\
&= \sec x \cdot \tan x.
\end{aligned}$$

■

Definition 2.7.6 — The Derivative of Cosecant. If $f(x) = \csc x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x \cot x$$

Proof. As with the secant function, we first note simply that $\csc x = \frac{1}{\sin x}$ by definition. We then apply the Quotient Rule with $f(x) = 1$ and $g(x) = \sin x$, again making use of simple trigonometric quotient and reciprocal identities.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\csc x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sin x}\right] \\
&= \frac{(\sin x)[1]' - (1)[\sin x]'}{\sin^2 x} \\
&= \frac{0 - \cos x}{\sin^2 x} \\
&= \frac{-\cos x}{\sin^2 x} \\
&= \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin x} \\
&= \cot x \cdot -\csc x \\
&= -\csc x \cdot \cot x.
\end{aligned}$$

■

2.8 Derivatives of Exponential and Logarithmic Functions

Since we've established derivatives for monomial, polynomial, rational, and trigonometric functions, that leaves only the exponential and logarithmic functions.

2.8.1 Derivative of the Exponential

First, we need a useful limit that will help us find the derivative of $f(x) = e^x$.

Note 2.8.1 — A Useful Limit. We note that

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

Definition 2.8.1 — The Derivative of the Natural Exponential. If $f(x) = e^x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x.$$

Proof. By definition,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Therefore, if $f(x) = e^x$, $f(x+h) = e^{x+h}$ and

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}.$$

Using properties of exponents to write $e^{x+h} = e^x e^h$,

$$f'(x) = \lim_{h\to 0} \frac{e^x e^h - e^x}{h}.$$

Then we can factor an e^x out of the numerator and out of the limit to obtain

$$f'(x) = e^x \lim_{h\to 0} \frac{e^h - 1}{h}.$$

Using note 2.8.1, we have that $f'(x) = e^x(1)$, and therefore

$$f'(x) = e^x.$$

■

Through a similar means, we can find the derivative of exponential with a base other than e.

Definition 2.8.2 — The Derivative of the Exponential. If $f(x) = b^x$ with $b > 0$ and $b \neq 1$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[b^x] = b^x \cdot \ln(b).$$

2.8.2 Derivative of the Logarithm

The proof for the derivative of the logarithm involves the definition of the natural base e as

$$e = \lim_{h\to 0} (1+h)^{\frac{1}{h}}.$$

It then involves some change of variables and clever algebraic manipulations, but the punchline of the proof is easy enough to get via another, more clever approach. First, we need to recall that $g(x)$ is an inverse of $f(x)$ if and only if $f(g(x)) = x = g(f(x))$. Now, we recall that the derivative of the function $h(x) = x$ is $h'(x) = 1$, so for $h(x) = f(g(x)) = x$, the derivative is $(f(g(x)))' = 1$. We then recall that $(f(g(x)))' = f'(g(x))g'(x)$ by the Chain Rule and obtain that

$$(f(g(x)))' = f'(g(x))g'(x) = 1,$$

and taking $f'(g(x))g'(x) = 1$, we can rearrange it into the following useful theorem, for which the preceding text is a decent sketch of a proof.

Theorem 2.8.1 For differentiable inverse functions f and g on (a,b) with $g'(x) \neq 0$ for all $x \in (a,b)$, if $x \in f((a,b))$, then

$$g'(x) = \frac{1}{f'(g(x))}.$$

Then using Theorem 2.8.1 with $f(x) = e^x$ and $g(x) = \ln x$, we have that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{e^{g(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

We sum this up neatly with the following logic:

Definition 2.8.3 — The Derivative of the Natural Logarithm. If $f(x) = \ln x$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}.$$

Via similar methods, we can arrive at the derivative for bases other than e, namely the following:

Definition 2.8.4 — The Derivative of the Logarithm. If $f(x) = \log_b(x)$ with $b > 0$ and $b \neq 1$, then

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[\log_b(x)] = \frac{1}{x\ln(b)}.$$

2.9 The Chain Rule

Definition 2.9.1 — The Chain Rule - Lagrange Notation. Given functions $f(x)$ and $g(x)$ such that $h(x) = f(g(x))$ is well-defined, we have that

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Definition 2.9.2 — The Chain Rule - Leibniz Notation. Given functions f and g such that $h = f \circ g$ is well-defined, we have that

$$\frac{\mathrm{d}h}{\mathrm{d}x} = \frac{\mathrm{d}h}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}.$$

Proof. While the traditional proof of the Chain Rule is not beyond the scope of this course, it is so mired in minute bookkeeping and arcane change-of-variables that its inclusion here would help no one, and is thus omitted. ■

> **Note 2.9.1 — Applying the Chain Rule.** Either way you state the Chain Rule, it is used the same. To take the derivative of a composition, take the derivative of the outermost function first, plugging in every other function that was already inside it. Then, if what we plugged in was more than just the identity function x, we must multiply by its derivative.

■ **Example 2.15** Let $h(x) = \sin(\cos(x^2))$. Find $h'(x)$.

Using note 2.9.1 above, we note that $h(x)$ is clearly a function composition, so the Chain Rule is the rule to use here. We therefore note that the sine function is the outermost function, so we take its derivative, which we know to be the cosine function. We then plug back in everything else that was on the inside (which was $\cos(x^2)$,) to get the first part of the derivative, which is $\cos(\cos(x^2))$.

Since the $\cos(x^2)$ we plugged back in is more complicated than just x, we must multiply this by the derivative of what we plugged in. We therefore have that

$$h'(x) = \cos(\cos(x^2)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}[\cos(x^2)].$$

We still have a derivative to take, and we note that the remaining $\cos(x^2)$ is another function composition, so the Chain Rule therefore applies again. So we take the derivative of the outer cosine function and plug the x^2 into the derivative we found (which is negative sine). Since x^2 is more complicated than x, we have to multiply by its derivative. We carry down everything else we've already done so then we have that

$$h'(x) = \cos(\cos(x^2)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}[\cos(x^2)] = \cos(\cos(x^2)) \cdot \left[-\sin(x^2) \cdot \frac{\mathrm{d}}{\mathrm{d}x}[x^2]\right].$$

All that is left now is to take the derivative of x^2, which is clearly $2x$. Since the function we squared was just x, we are done, though it is of note that you are welcome to take and multiply by the derivative of x if you wish (since it is just 1).

$$h'(x) = \cos(\cos(x^2)) \cdot [-\sin(x^2) \cdot 2x].$$

The last step is to rearrange things. The preferred way is constants first, then increasingly complex expressions as you read left to right, as such

$$h'(x) = -2x\sin(x^2)\cos(\cos(x^2)).$$

■

Since the Chain Rule is without a doubt the most important and most convoluted of the rules, what follows are several examples, using every different type of derivative we have thusfar studied. While reading through these, it is advisable that the list of derivatives be kept handy!

■ **Example 2.16** Find $f'(x)$ if $f(x) = [\ln(\csc x)]$

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}[\ln(\csc x)] &= \tfrac{1}{\csc x} \cdot \tfrac{d}{dx}[\csc x] \\
&= \tfrac{1}{\csc x} \cdot -\csc x \cot x \\
&= -\cot x.
\end{aligned}$$

■

■ **Example 2.17** Find $\frac{d}{dx}[\sin x \cdot \ln[\cos(x)]]$.

$$\begin{aligned}
\frac{d}{dx}[\sin x \cdot \ln[\cos(x)]] &= \cos x \cdot \ln(\cos x) + \sin x \frac{d}{dx}[\ln(\cos(x))] \\
&= \cos x \cdot \ln(\cos x) + \sin x[\tfrac{1}{\cos x} \cdot -\sin x] \\
&= \cos x \cdot \ln(\cos x) - \sin x \tan x.
\end{aligned}$$

■

■ **Example 2.18** Find $\frac{d}{dx}[\ln(\frac{3}{x})]$.

$$\begin{aligned}
\frac{d}{dx}[\ln(\tfrac{3}{x})] &= \frac{1}{\frac{3}{x}} \cdot \frac{d}{dx}[\tfrac{3}{x}] \\
&= \tfrac{x}{3} \cdot \frac{d}{dx}[3x^{-1}] \\
&= \tfrac{x}{3} \cdot [-3x^{-2}] \\
&= x \cdot -x^{-2} \\
&= -\tfrac{x}{x^2} \\
&= -\tfrac{1}{x}.
\end{aligned}$$

■

■ **Example 2.19** Find $\frac{d}{dx}[3^x]$.

$$\frac{d}{dx}[3^x] = 3^x \cdot \ln(3).$$

■

■ **Example 2.20** Find $\frac{d}{dx}[\log_{12} x]$.

$$\frac{d}{dx}[\log_{12} x] = \frac{1}{x \cdot \ln(12)}.$$

■

■ **Example 2.21** Find $\frac{d}{dx}[(\sin x)^2]$.

$$\begin{aligned}
\frac{d}{dx}[(\sin x)^2] &= 2\sin x \cdot \frac{d}{dx}[\sin x] \\
&= 2\sin x \cdot \cos x.
\end{aligned}$$

■

■ **Example 2.22** Find $\frac{d}{dx}[\sin(\cos x)]$.

$$\begin{aligned}
\frac{d}{dx}\sin(\cos x) &= \cos(\cos x) \cdot \frac{d}{dx}[\cos x] \\
&= \cos(\cos x) \cdot -\sin x \\
&= -\sin x \cdot \cos(\cos x).
\end{aligned}$$

■

■ **Example 2.23** Find $\frac{d}{dx}\left[\frac{x^2+1}{x-3}\right]$.

Notice that $\frac{x^2+1}{x-3} = (x^2+1)(x-3)^{-1}$. Therefore,

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^2+1}{x-3}\right] &= \frac{d}{dx}[(x^2+1)(x-3)^{-1}] \\
&= [x^2+1]'(x-3)^{-1} + (x^2)[(x-3)^{-1}]' \\
&= [2x](x-3)^{-1} + (x^2+1)[-(x-3)^{-2}] \\
&= \frac{2x}{x-3} - \frac{x^2+1}{(x-3)^2}.
\end{aligned}$$

■

■ **Example 2.24** Find $\frac{d}{dx}[\sin(\cos(\tan x))]$.

$$\begin{aligned}
\tfrac{d}{dx}[\sin(\cos(\tan x))] &= \cos(\cos(\tan x)) \cdot \tfrac{d}{dx}[\cos(\tan x)] \\
&= \cos(\cos(\tan x)) \cdot -\sin(\tan x) \cdot \tfrac{d}{dx}(\tan x) \\
&= \cos(\cos(\tan x)) \cdot -\sin(\tan x) \cdot \sec^2 x \\
&= -\sec^2 x \cdot \sin(\tan x) \cdot \cos(\cos(\tan x)).
\end{aligned}$$

■

■ **Example 2.25** Find $\frac{d}{dx}[\sin(\tan(x^2))]$.

$$\begin{aligned}
\tfrac{d}{dx}[\sin(\tan(x^2))] &= \tfrac{d}{dx}[\sin(\tan(x^2))] \\
&= \cos(\tan(x^2)) \cdot \sec^2(x^2) \cdot 2x \\
&= 2x \cdot \sec^2(x^2) \cdot \cos(\tan(x^2)).
\end{aligned}$$

■

■ **Example 2.26** Find $\frac{d}{dx}[\tan(\sin(\cos(x)))]$.

$$\begin{aligned}
\tfrac{d}{dx}[\tan(\sin(\cos(x)))] &= \sec^2(\sin(\cos x)) \cdot \cos(\cos x) \cdot -\sin x \\
&= -\sin x \cdot \cos(\cos(x)) \cdot \sec^2 x(\sin(\cos x)).
\end{aligned}$$

■

■ **Example 2.27** Find $\frac{d}{dx}[\sqrt{\sin(\csc(x))}]$.

$$\begin{aligned}
\tfrac{d}{dx}[\sqrt{\sin(\csc(x))}] &= \tfrac{d}{dx}[(\sin(\csc x))^{1/2}] \\
&= \tfrac{1}{2}(\sin(\csc x))^{-\frac{1}{2}} \cdot \cos(\csc x) \cdot -\csc x \cot x \\
&= -\tfrac{1}{2}(\sin(\csc x))^{-\frac{1}{2}} \cdot \csc x \cot x \cdot \cos(\csc x).
\end{aligned}$$

■

■ **Example 2.28** Find $\frac{d}{dx}[\csc(\sin x)]$.

$$\begin{aligned}
\tfrac{d}{dx}[\csc(\sin x)] &= -\csc(\sin x)\cot(\sin x) \cdot \tfrac{d}{dx}[\sin x] \\
&= -\cos x \cdot \csc(\sin x)\cot(\sin x).
\end{aligned}$$

■

■ **Example 2.29** Find $\frac{d}{dx}[3e^{2x}]$.

$$\begin{aligned}
\tfrac{d}{dx}[3e^{2x}] &= 3\tfrac{d}{dx}[e^{2x}] \\
&= 3[e^{2x} \cdot \tfrac{d}{dx}[2x]] \\
&= 3[2e^{2x}] \\
&= 6e^{2x}.
\end{aligned}$$

■

■ **Example 2.30** Find $\frac{d}{dt}[e^{-3t}]$.

$$\begin{aligned}
\tfrac{d}{dt}[e^{-3t}] &= e^{-3t} \cdot \tfrac{d}{dt}[-3t] \\
&= -3e^{-3t}.
\end{aligned}$$

■

■ **Example 2.31** Find $\frac{\mathrm{d}}{\mathrm{d}\theta}[e^{\sin\theta}]$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}[e^{\sin\theta}] &= e^{\sin\theta}\cdot\frac{\mathrm{d}}{\mathrm{d}x}[\sin\theta]\\ &= e^{\sin\theta}\cdot\cos\theta\\ &= \cos\theta e^{\sin\theta}.\end{aligned}$$

■

■ **Example 2.32** Find $\frac{\mathrm{d}}{\mathrm{d}z}[\frac{1}{e^{3z}}]$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}[\tfrac{1}{e^{3z}}] &= \frac{\mathrm{d}}{\mathrm{d}z}[1\cdot e^{-3z}]\\ &= e^{-3z}\cdot\frac{\mathrm{d}}{\mathrm{d}z}[-3z]\\ &= -3e^{-3z}\\ &= -\frac{3}{e^{3z}}.\end{aligned}$$

■

■ **Example 2.33** Find $f'(x)$ if $f(x)=\ln(x^2)$.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\ln(x^2)] &= \frac{1}{x^2}\cdot\frac{d}{dx}[x^2]\\ &= \frac{1}{x^2}\cdot 2x\\ &= \frac{2x}{x^2}\\ &= \frac{2}{x}.\end{aligned}$$

■

Definition 2.9.3 — Derivative of a Logarithmic Composition. If $f(x)$ is a function with a domain such that $f(x)\neq 0$, then we note that

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln|f(x)|]=\frac{1}{f(x)}\cdot f'(x)=\frac{f'(x)}{f(x)}.$$

2.10 Derivatives of the Inverse Trigonometric Functions

Recall Theorem 2.8.1, which stated for differentiable inverse functions f and g on (a,b) with $g'(x)\neq 0$ for all $x\in(a,b)$, if $x\in f((a,b))$, then

$$g'(x)=\frac{1}{f'(g(x))}.$$

Using this together with the definition of an inverse function and a bit of trigonometry, we will find the derivatives of both the inverse sine and inverse tangent functions.

2.10.1 Derivative of the Inverse Sine and Cosine Functions

Recall the definition of the inverse sine function, sometimes denoted $\sin^{-1}(x)$, but here denoted $\arcsin x$:

Definition 2.10.1 For $-\frac{\pi}{2}\leq y\leq\frac{\pi}{2}$, $y=\arcsin x\Leftrightarrow\sin y=x$.

Note 2.10.1 We restrict y to obtain some level of normalcy and consistency out of the inverse sine function. In fact, the inverse sine isn't even a function without a restriction (as it has multiple outputs for the same input). Also, since $\sin y\in[-1,1]$, we have that $x\in[-1,1]$.

■

Since the sine and inverse sine functions are inverses, by the definition of an inverse we have that

$$\arcsin(\sin(x)) = x = \sin(\arcsin(x)),$$

for the proper restrictions of the domain of $\arcsin x$, as usual. Then to find the derivative of the inverse sine function, we will use Theorem 2.8.1 with $f(x) = \sin x$ and $g(x) = \arcsin x$. Then we have that $f'(x) = \cos x$ and it follows that

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)},$$

which is not at all helpful. The problem in this case is that $\arcsin x$ composed inside the cosine function, so simplifying that would be helpful. Recall from Definition 2.10.1 that $y = \arcsin x$. Then we write y instead of $\arcsin x$ and our formula becomes

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\cos(y)}.$$

Then by the Pythagorean Identity, we know that $\cos^2 y + \sin^2 y = 1$, and solving for $\cos y$ gives us $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-(\sin y)^2}$. Since $x = \sin y$ when $y = \arcsin x$, we replace the $\sin y$ with x in the denominator of our function. Then we have found out derivative, namely

Definition 2.10.2 — The Derivative of the Inverse Sine. Given the inverse sine function $g(x) = \arcsin x$, with $g(x) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$g'(x) = \frac{1}{\sqrt{1-x^2}}.$$

We also give the derivative of the inverse cosine function, namely

Definition 2.10.3 — The Derivative of the Inverse Cosine. Given the inverse cosine function $g(x) = \arccos x$, with $g(x) \in [0, \pi]$,

$$g'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

Exercise 2.13 Using the same logic as the derivative for inverse sine, show how to derive the inverse cosine function. ■

2.10.2 Derivative of the Other Inverse Trigonometric Functions

The same logic that was used to derive the derivative of the inverse sine function can be applied to the remaining inverse trigonometric functions, and their derivatives are summarized below.

$$\begin{array}{lcl lcl}
\frac{d}{dx}(\arcsin x) & = & \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}(\arccos x) & = & -\frac{1}{\sqrt{1-x^2}} \\
\frac{d}{dx}(\operatorname{arccsc} x) & = & -\frac{1}{|x|\sqrt{x^2-1}} & \frac{d}{dx}(\operatorname{arcsec} x) & = & \frac{1}{|x|\sqrt{x^2-1}} \\
\frac{d}{dx}(\arctan x) & = & \frac{1}{1+x^2} & \frac{d}{dx}(\operatorname{arccot} x) & = & -\frac{1}{1+x^2}
\end{array}$$

Figure 2.10: A table of inverse trigonometric derivatives.

2.11 The Hyperbolic Functions

There exists a set of functions that show up enough in physical applications that they are given a special name. In the same way that the usual trigonometric functions $f(\theta) = \cos\theta$, $f(\theta) = \sin\theta$, and so on are defined so that all points $(\cos\theta, \sin\theta)$ for $0 \le \theta < 2\pi$ form the unit circle, the hyperbolic points $(\cosh t, \sinh t)$ form the right half of what is called the unit hyperbola.

Any application of such a function includes detail about it far beyond the scope of this text, so we will provide a few definitions and then derive a derivative, and end by offering another list of six derivatives for the overburdened reader to memorize.

Definition 2.11.1 — The Hyperbolic Functions. For all $x \in \mathbb{R}$, the hyperbolic trigonometric functions are defined as

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} & \cosh x &= \frac{e^x + e^{-x}}{2} \\ \tanh x &= \frac{\sinh x}{\cosh x} & \coth x &= \frac{\cosh x}{\sinh x} \\ \operatorname{sech} x &= \frac{1}{\cosh x} & \operatorname{csch} x &= \frac{1}{\sinh x} \end{aligned}$$

■ **Example 2.34** If $f(x) = \sinh x$, say we were asked to find $f'(x)$. This is easy enough. First, we note that

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{2}(e^x - e^{-x}).$$

Then we have that

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}(e^x - e^{-x})\right] = \frac{1}{2}\left(\frac{\mathrm{d}}{\mathrm{d}x}(e^x) - \frac{\mathrm{d}}{\mathrm{d}x}(e^{-x})\right) = \frac{1}{2}\left((e^x) - (-e^{-x})\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

■

Exercise 2.14 Find $f'(x)$ for $f(x) = \cosh x$. ■

2.11.1 Derivatives of the Hyperbolic Functions

We now state the rest of the derivatives of the hyperbolic functions. You may notice some striking similarities and subtle differences when comparing these to the derivatives of the usual trigonometric derivatives.

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}x}(\sinh x) &= \cosh x & \tfrac{\mathrm{d}}{\mathrm{d}x}(\cosh x) &= \sinh x \\ \tfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{csch} x) &= -\operatorname{sech} x \tanh x & \tfrac{\mathrm{d}}{\mathrm{d}x}(\operatorname{sech} x) &= -\operatorname{csch} x \coth x \\ \tfrac{\mathrm{d}}{\mathrm{d}x}(\tanh x) &= \operatorname{sech}^2 x & \tfrac{\mathrm{d}}{\mathrm{d}x}(\coth x) &= -\operatorname{csch}^2 x \end{aligned}$$

Figure 2.11: A table of hyperbolic trigonometric derivatives.

■ **Example 2.35** If $h(x) = e^x \sinh x$, find $h'(x)$. First, we note this is a product of two functions. We will therefore use the product rule together with the derivative of $\sinh x$ from Table 2.11 to obtain that

$$h'(x) = \frac{\mathrm{d}}{\mathrm{d}x}[e^x](\sinh x) + (e^x)\frac{\mathrm{d}}{\mathrm{d}x}[\sinh x] = e^x \sinh x + e^x \cosh x = e^x(\sinh x + \cosh x).$$

■

Exercise 2.15 Find $\frac{\mathrm{d}}{\mathrm{d}x}[e^x \cosh x]$. ■

Exercise 2.16 Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\cosh^2 x - \tan x\right]$. ■

Exercise 2.17 Find $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2-4x+1}{\tanh x}\right]$. ■

Exercise 2.18 Find $\frac{\mathrm{d}}{\mathrm{d}x}[\operatorname{sech}(\sinh(x))]$. ■

2.12 The Implicit Derivative

Definition 2.12.1 — Implicit Equation in Two Variables. An implicit equation in two variables is a relation of the form $R(x,y) = 0$. The most notable implicit equation we know is that of the Unit Circle, namely

$$x^2 + y^2 - 1 = 0.$$

A few other examples of implicit functions include $x^2 + y^2 = 1$, $xy + 1 = 4$ and $\sin(xy) = \frac{1}{x}$. Note that while each of these do not meet the criteria of being a relation set equal to zero, it is simply a matter of subtracting the right-hand side from both sides of the equation to do so!

Note 2.12.1 When considering implicit equations or functions, keep in mind that y is in fact $y(x)$. That is, y is a function of x.

Definition 2.12.2 — The Implicit Derivative. Given a relation $R(x,y) = 0$, the implicit derivative is a function of the form

$$y'(x) = f(x,y).$$

We usually omit the 'of x' part of $y'(x)$, just writing instead

$$y' = f(x,y).$$

Note that the implicit derivative is written as y' equals some function of x and y, which means that there should be no y' terms on the right-hand side of the equation when it is solved!

■ **Example 2.36** Find the implicit derivative of $x^2 + y^2 = 1$.

1. Take the derivative with respect to x of both sides, using properties of derivatives when needed. Remember that y is a FUNCTION of x, and therefore the Chain Rule must be applied

to it:

$$\begin{aligned}
\frac{d}{dx}[x^2+y^2] &= \frac{d}{dx}[1] \\
\frac{d}{dx}[x^2]+\frac{d}{dx}[y^2] &= 0 \\
2x+\frac{d}{dx}[y(x)^2] &= 0 \\
2x+2(y(x))^{2-1}\cdot y'(x) &= 0 \\
2yy'+2x &= 0.
\end{aligned}$$

2. We now simply solve for $y'(x)$:

$$\begin{aligned}
2yy' &= -2x \\
y' &= \frac{-2x}{2y} \\
y' &= \frac{-x}{y}.
\end{aligned}$$

■

In this next example, we see what happens if we have a y on each side of the equation. In that case, it is simply a matter of taking the derivative of both sides, remembering the Chain Rule, then going through the algebraic manipulations in order to get the function y' by itself. As usual, the hardest part of this is the algebra, not the calculus!

■ **Example 2.37** Find y' if $x^3y-x=y$.

$$\begin{aligned}
\frac{d}{dx}[x^3y-x] &= \frac{d}{dx}[y] \\
\frac{d}{dx}[x^3y]-\frac{d}{dx}[x] &= y' \\
3x^2\cdot(y)+(x^3)[y']-1 &= y' \\
3x^2y+x^3y'-1 &= y' \\
x^3y'-y' &= -3x^2y+1 \\
y'(x^3-1) &= -3x^2y+1 \\
y' &= \frac{-3x^2y+1}{x^3-1}.
\end{aligned}$$

■

Finally, we will look at one final example, which is about as nasty as they get. Same process, namely

Note 2.12.2 — The Process for Finding Implicit Derivatives. The process for finding implicit derivatives is always the same, namely

1. Take the derivative of both sides.
2. Use the laws of derivatives carefully, remembering that y is a function of x and therefore its derivative requires the Chain Rule.
3. Move all terms with y' to the left-hand side.
4. Move all terms without a y' to the right-hand side.
5. Factor out a y' on the left-hand side.
6. Divide both sides by what is left after y' is factored.
7. Simplify the right-hand side, if possible.

■ **Example 2.38** Find y' if $\frac{y-1}{x-1} = \cos(xy)$.

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}x}\left[\tfrac{y-1}{x-1}\right] &= \tfrac{\mathrm{d}}{\mathrm{d}x}[\cos(xy)] \\
\tfrac{\mathrm{d}}{\mathrm{d}x}\left[(y-1)(x-1)^{-1}\right] &= -\sin(xy)\cdot\tfrac{\mathrm{d}}{\mathrm{d}x}[xy] \\
y'(x-1)^{-1}+(y-1)\tfrac{\mathrm{d}}{\mathrm{d}x}\left[(x-1)^{-1}\right] &= -\sin(xy)\cdot[[1](y)+(x)[y']] \\
y'(x-1)^{-1}+(y-1)\left[-1(x-1)^{-2}\cdot 1\right] &= -y\sin(xy)-xy'\sin(xy) \\
y'(x-1)^{-1}+xy'\sin(xy) &= -y\sin(xy)+(y-1)\left[-(x-1)^{-2}\right] \\
y'\left[(x-1)^{-1}+x\sin(xy)\right] &= -y\sin(xy)+(y-1)(x-1)^{-2} \\
y' &= \tfrac{y\sin(xy)+(y-1)(x-1)^{-2}}{(x-1)^{-1}+y\sin(xy)}.
\end{aligned}$$

■

2.13 Proofs of Advanced Derivative Rules

As promised in earlier sections, we will now prove the Power, Product, and Quotient Rules.

Proof of the Power Rule

Recall the Power Rule:

Theorem 2.13.1 — The Power Rule, Lagrange Notation. For $n \in \mathbb{R}$ and $f(x) = x^n$,

$$f'(x) = nx^{n-1}.$$

Proof. Let $n \in \mathbb{R}$ and define $f(x) = y = x^n$. Then

$$\begin{aligned}
\ln y &= \ln x^n \\
\ln y &= n\ln x \\
\tfrac{\mathrm{d}}{\mathrm{d}x}[\ln y] &= \tfrac{\mathrm{d}}{\mathrm{d}x}[n\ln x] \\
\frac{y'}{y} &= n\frac{1}{x}.
\end{aligned}$$

Then solving for y' gives

$$y' = y\frac{n}{x}.$$

But $f(x) = y = x^n$, and therefore $f'(x) = y' = \dfrac{nx^n}{x} = nx^{n-1}$ as required. ■

Proof of the Product Rule

Now we prove the Product Rule. Let us remember:

Theorem 2.13.2 — The Product Rule - Lagrange Notation. For differentiable functions f and g,

$$[f(x)\cdot g(x)]' = [f'(x)](g(x)) + (f(x))[g'(x)].$$

Proof. Let f and g be differentiable functions and let $f(x) = y = fg$. Then

$$\ln y = \ln(fg) = \ln(f) + \ln(g).$$

Then if we take the derivative of both sides of the equation, we have that

$$\frac{y'}{y} = \frac{f'}{f} + \frac{g'}{g}.$$

Solving for y' gives us that

$$y' = y\left(\frac{f'}{f} + \frac{g'}{g}\right).$$

Then since $f(x) = y = fg$, we have that

$$f'(x) = y' = (fg)\left(\frac{f'}{f} + \frac{g'}{g}\right) = \frac{f'fg}{f} + \frac{g'fg}{g} = f'g + g'f.$$

Then $f'(x) = f'g + fg'$ as required. ■

Proof of the Quotient Rule

Finally, we prove the Quotient Rule. Let us recall:

Definition 2.13.1 — The Quotient Rule - Lagrange Notation. Let f and g be differentiable functions such that $g \neq 0$. Then

$$\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}.$$

Proof. Let f and g be differentiable functions such that $g \neq 0$. Let $h = \frac{f}{g}$ and note that $h = fg^{-1}$. Then by the Product Rule, we have that

$$h' = \frac{\mathrm{d}}{\mathrm{d}x}(fg^{-1}) = [f'](g^{-1}) + (f)\frac{\mathrm{d}}{\mathrm{d}x}[g^{-1}].$$

We note that the Chain Rule is needed for $\frac{\mathrm{d}}{\mathrm{d}x}[g^{-1}]$, namely

$$\frac{\mathrm{d}}{\mathrm{d}x}[g^{-1}] = -g^{-1-1} \cdot g' = -g^{-2}g'.$$

Then we have that

$$h' = \frac{\mathrm{d}}{\mathrm{d}x}(fg^{-1}) = [f'](g^{-1}) + (f)[-g^{-2}g'] = \frac{f'}{g} + \frac{fg'}{g^2}.$$

Then getting a common denominator gives us that

$$h' = \frac{f'}{g} - \frac{fg'}{g^2} = \left(\frac{g}{g}\right)\frac{f'}{g} - \frac{fg'}{g^2} = \frac{f'g - g'f}{g^2}.$$

This concludes the proof. ■

3. Advanced Topics in Derivatives

Finally, we've arrived at the long-awaited answer to 'what is Calculus used for?' We only cover a small number of applications. The first application is in Physics, where we relate an object's position, velocity, speed, and acceleration to each other using the derivative. In the second application, we discuss an example of how Calculus is used in the social sciences. The third and final application is about related rates problems where ladders are falling down walls and spheres are being filled with air. Then we end by discussing a theoretical application of the derivative that helps us find limits, something we thought we were done with! Just for our knowledge, we're never done with limits. They are here to stay.

3.1 The Derivative as Rate of Change

Let's begin by defining the functions $s(t)$, $v(t)$, and $a(t)$ as such:

Definition 3.1.1 The position function $s(t)$ gives an object's position at time t. The velocity function $v(t)$ gives an object's velocity at time t. The acceleration function $a(t)$ gives an object's acceleration at time t.

The first question that is usually asked is why position is denoted $s(t)$ instead of $p(t)$, as we may suppose. As usual, we have the Germans to blame for this. The German word '*streke*' means 'distance' in English, and it starts with the s. The actual term in physics is 'displacement,' but we will use position for the sake of simplicity in our text.

The second question is usually asking about the difference between speed and velocity. That is simple. First, we have to define velocity:

Note 3.1.1 Velocity is the rate of change of position, that is

$$v(t) = s'(t)$$

or

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}.$$

Since we know that $v(t)$ is the rate of change of position, we can get speed by taking the absolute value $|v(t)|$. That is, speed is always positive. If we were to back our car into a badly-placed moose at -50mph, our speedometer would read 50mph when we did it. The velocity being negative simply tells us we were going in *reverse*.

The final question involves the relationship between acceleration and velocity. Mathematically speaking, it is as follows:

Note 3.1.2 Acceleration is the rate of change of velocity, that is

$$a(t) = v'(t) = s''(t)$$

or

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 v}{\mathrm{d}t^2}.$$

Simple enough, but words get in the way here. Acceleration is the rate of change of velocity. Velocity is the rate of change of position. Therefore acceleration is the rate of change of the rate of change of position. Another way to say it (using the example of a car) is that velocity is how fast the car is moving (in one direction or another,) and acceleration is how fast the car is changing its velocity.

Keep in mind that since we're doing real world examples, we will need units on our measurements. Units of distance are for position (so feet (ft) and meters (m)). Velocity is measured in units of distance per unit of time (so feet per second ($^{\mathrm{ft}}/_{\mathrm{s}}$) and meters per second ($^{\mathrm{m}}/_{\mathrm{s}}$)). Acceleration is the rate of change of velocity per second, which means that acceleration is measured in units of distance per second *per second*. That is units of distance per second squared, since for d some distance

$$\frac{^{\mathrm{d}}/_{\mathrm{s}}}{s} = \frac{^{\mathrm{d}}/_{\mathrm{s}}}{^{s}/_{1}} = \frac{\mathrm{d}}{\mathrm{s}} \cdot \frac{1}{s} = \frac{d}{s \cdot s} = \frac{d}{s^2}.$$

Then acceleration is usually measured in feet per second squared ($^{\mathrm{ft}}/_{\mathrm{s}^2}$) or meters per second squared ($^{\mathrm{m}}/_{\mathrm{s}^2}$). Of course, there are several other ways to measure all three of these, the most common of which being position in miles (mi), velocity in miles per hour (mph or $^{\mathrm{mi}}/_{\mathrm{h}}$), and acceleration, which usually still uses the examples given above.

■ **Example 3.1** Suppose there is a car with position (in meters) given by the function $s(t) = 2t^3 - 4t^2 - 6t + 1$. At what time t (in seconds) is the car speeding up or slowing down?

As with any problem in mathematics, we begin with what we know. We know that $s(t) = 2t^3 - 4t^2 - 6t + 1$. We also know that the question is asking about speeding up and slowing down, which is a question about acceleration. Therefore we must find $a(t)$ given the relationships above. Since we know that velocity is the derivative (or rate of change) of position ($s'(t) = v(t)$) and that acceleration is the derivative (or rate of change) of velocity ($v'(t) = a(t)$), we must find $v(t)$ and then $a(t)$ as such:

$$s'(t) = v(t) = 6t^2 - 8t - 6$$

so that

$$v'(t) = a(t) = 12t - 8.$$

So the question asks 'at what time is the car speeding up or slowing down?' We must therefore find out whether acceleration is positive or negative. If the acceleration is positive ($a(t) > 0$), the car is speeding up. If the acceleration is negative ($a(t) < 0$), the car is slowing down.

So we find $a(t) > 0$ with some trivial algebra

$$\begin{aligned} a(t) &> 0 \\ 12t - 8 &> 0 \\ 12t &> 8 \\ t &> 8/12 \\ t &> 2/3 \text{ seconds.} \end{aligned}$$

In the same way, we find $a(t) < 0$

$$\begin{aligned} a(t) &< 0 \\ 12t - 8 &< 0 \\ 12t &< 8 \\ t &< 8/12 \\ t &< 2/3 \text{ seconds.} \end{aligned}$$

Since negative time is meaningless in this example, we have that between 0 and $\frac{2}{3}$ seconds, the car is slowing down. After that, it continuously speeds up. ■

■ **Example 3.2** A toy rocket is shot into the air. The position in feet at time t (in seconds) is $s(t) = -16t^2 + 200t + 10$.

1. When is the rocket at the apex of its flight?
2. At the apex, how high is the rocket in the air?
3. When does the rocket land?

1. To find when the toy rocket is at the apex (or top) of its flight, we have to consider its velocity. If we think of any object hurled into the air, it begins decelerating the moment it leaves the surface of earth due to gravity. As time moves forward and the object travels upwards, it has a decreasing positive velocity. At some point during its course, its velocity is zero before it becomes an increasing negative velocity due to gravity pulling down. It should make sense then that the top of the rocket's arc is going to be when $v(t) = 0$. We therefore must first find $v(t)$, by noting that $v(t) = s'(t)$. So we have

$$\begin{aligned} v(t) &= s'(t) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}[-16t^2 + 200t + 10] \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}[-16t^2] + \tfrac{\mathrm{d}}{\mathrm{d}t}[200t] + \tfrac{\mathrm{d}}{\mathrm{d}t}[10] \\ &= -32t + 200. \end{aligned}$$

So now we must find $v(t) = 0$ or

$$
\begin{aligned}
-32t+200 &= 0\\
-32t &= -200\\
t &= {}^{-200}\!/\!_{-32}\\
t &= 6.25\,\text{seconds.}
\end{aligned}
$$

So the toy rocket reaches its apex at time $t = 6.25$ seconds.

2. The second questions asks how high is the toy rocket in the air at the apex? Since we just found that $t = 6.25$ is the time of apex, we note that the question asks about position. Therefore we find $s(6.25)$ as follows:

$$
\begin{aligned}
s(6.25) &= -16(6.25)^2+200(6.25)+10\\
&= -625+1250+10\\
&= 635\,\text{ft}
\end{aligned}
$$

3. The third and final question asks when the rocket lands. If we take a moment and think about this, it is asking when the position of the rocket is the same as when it started. We must therefore find when $s(t) = 0$, which is a quadratic equal to zero. We will use the quadratic formula, which looks like

$$
\begin{aligned}
s(t) &= 0\\
-16t^2+200t+10 &= 0\\
t &= \frac{-200\pm\sqrt{(200)^2-4(-16)(10)}}{2(-16)}\\
t &= \frac{-200\pm\sqrt{40000-(-6400)}}{-32}\\
t &= \frac{-200\pm\sqrt{46400}}{-32}\\
t &\approx \{-0.48\,\text{s}, 12.98\,\text{s}\}
\end{aligned}
$$

Obviously the toy rocket lands at the latest time, so it must land at $t = 12.98$ seconds. The other time is curious, however. Can we explain what it means? It must be the other time at which our position is at zero, but why might that be? It isn't a trick question, but here it is a question left to the reader.

■

Exercise 3.1 Explain how the negative answer in the previous example makes sense for being a time in which the position of the toy rocket is at zero. ■

3.2 Calculus in the Social Sciences

While there are many uses of concepts we have thusfar covered in the social sciences, the most obvious and straightforward is in the study of population growth.

There's a common worry about overpopulation that was popularized at the end of the 1960s by a biologist from Stanford named Paul Ehrlich. The common thread of logic was something to the effect of "if we keep making more people, we'll eventually run into resource scarcity and people will start dying off in droves." This point just before people start dying off in droves is called the carrying capacity, though it has a better definition.

Definition 3.2.1 — Carrying Capacity. The point at which a population stabilizes and is sustained by its environment is called the carrying capacity of that population.

To talk about population dynamics properly, we'd need the tools from differential equations, a field that often succeeds a Calculus II course. However, we have the tools to investigate the equation, so we'll do that, then state the closed-form solution (that is found using differential equation techniques), then use that to see when the world's population is going to stabilize.

3.2.1 The Logistic Equation

In general, a population grows at a rate proportional to its size. This should make some sense. If you consider putting a pair of rabbits of different biological sexes into a pen with enough food and water, they will multiply. If you put twenty pairs of such rabbits into the same situation, they will multiply faster.

We have this concept of the per capita growth rate, which is the ratio of the rate of change to the population. It is the constant k in the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP,$$

where k is the per capita growth rate, and P is a function of time t that represents the population at that time. We lack the tools at this point to properly do anything with this equation, but we can understand the left-hand side if we recall that $\mathrm{d}P$ and $\mathrm{d}t$ can be thought of in some sense as representing incredibly small changes in population and time, respectively.

Since they are in ratio to each other, the left-hand expression in the above equation is talking about how tiny changes in population over tiny changes in time is equivalent to the right side, which is a simple scalar multiple of the population.

If we introduce N as a related function of the carrying capacity, we have the logistic equation.

Definition 3.2.2 — The Logistic Equation. The logistic equation is a differential equation given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP(N-P).$$

Note 3.2.1 If $P=0$ or $P=N$, we note the above equation is in a state we call equilibrium. That is, the derivative expressed on the left-hand side is exactly zero everywhere. The equilibrium at $P=N$ is the carrying capacity in this case, while $P=0$ is the trivial equilibrium where there is simply no population in the first place!

It is possible to use the tools of differential equations to find a solution to the logistic equation.

Definition 3.2.3 — The Solution to the Logistic Equation. For k the per capita growth rate, P_0 the initial population, N the carrying capacity, and t the time (in years), the logistic equation has a closed-form solution of

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt}+1}.$$

Note 3.2.2 It is worth mentioning at this point that the scale for P_0 and N determine the scale for $P(t)$, and must be the same. In the example that follows, they are measured in billions.

■ **Example 3.3** If we look at estimations from leading population scientists, the world's population was $P_0 = 7.888$ billion in 2021. If we assume those same scientists are correct and that the per capita growth rate stays at $k = 0.002$ and we cap at $N = 10.2$ billion people, we can figure out when this will occur. Plugging in, we obtain

$$P(t) = \frac{10.2}{\frac{10.2-7.888}{7.888}e^{-(0.002)(10.2)t}+1} = \frac{10.2}{0.2931e^{-0.0204t}+1}.$$

This gives the following graph:

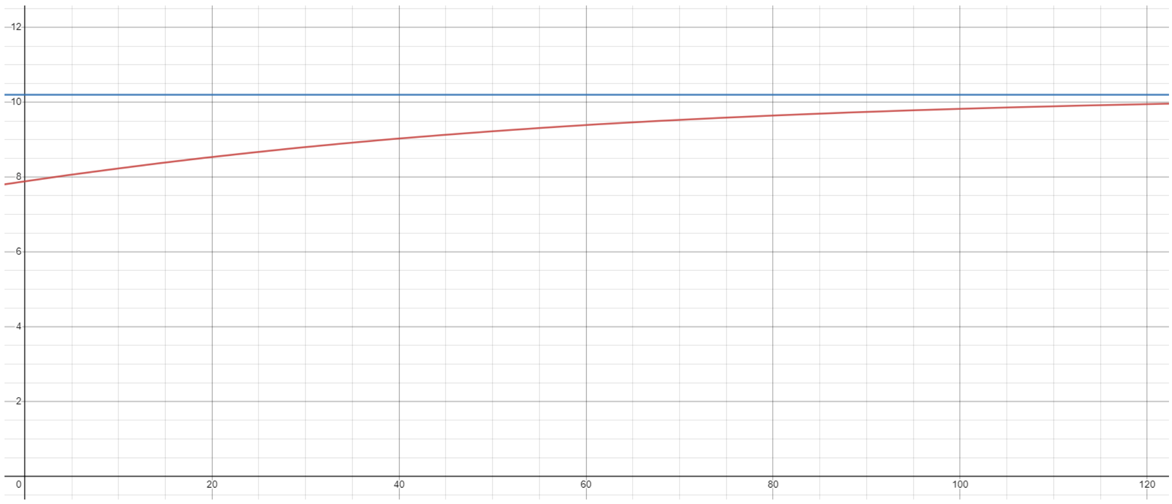

Figure 3.1: The world population reaching equilibrium.

As we can see, the population begins to level off somewhere around 2100, which is $t_0 + 79$ on this graph, where $t_0 = 2021$ from our data. ■

Exercise 3.2 How does changing the per capita growth rate to $k = 0.05$ change the graph from Figure 3.2? ■

3.3 Related Rates of Change

If we take a moment to think back to the initial definition of the derivative, we will recall that it measures the 'instantaneous rate of change' of a function at a point (in quotes because that phrase still does not quite make sense). This being the chapter that deals with applications of the derivative, let us now look at the concept of related rates with a few examples.

■ **Example 3.4** Suppose a ten-foot-ladder is placed against a six-foot-tall wall. It begins slipping down along the wall at a rate of 3 ft/s. When the top of the ladder is six feet above the ground, at what rate is the ladder moving away from the wall?

As is true with any problem in mathematics, it is usually a good strategy to write down what we know from the problem. We know the length l of the ladder is ten feet, so $l = 10$ft. We know that the height h of the wall is six feet, so $h = 6$ft. We know that the depth d of the ladder is *changing* at a rate of 3 ft/s, therefore $d' = 3$ft/s.

We know that the ladder is changing with respect to time in this problem, so both our d and h and l are all functions with respect to t. We also know the relationship (the 'related' part of related rates) between those three functions as

$$d^2 + h^2 = l^2,$$

since it is assumed the wall is built at a right angle.

What we are trying to find is the rate at which the height h is changing, and we measure rates of change by finding derivatives. Therefore we are trying to find h'. Notice, however, that while we have the value of d', or how the depth d of the ladder is changing, we do not have d itself! This is simple enough to remedy. From the Pythagorean relationship above,

$$d^2 + (6)^2 = (10)^2.$$

Solving this very simple equation gives us our d as

$$d = \sqrt{10^2 - 6^2} = \sqrt{100 - 36} = \sqrt{64} = 8.$$

If we take a moment to take stock of what we have and what we want, we may be able to move forward to find an answer. Omitting units for the sake of simplicity, we have that $l = 10$, $h = 6$, $d = 8$. For the related rates, we know that we need h' and have that $d' = 3$. Think for a moment about how we found the value of the depth d of the ladder. It was given to us in a relationship via the Pythagorean Theorem! Therefore to find our h' we need to find a relationship between it, d' and l'. The answer is simple, then. We will take the relationship and take the derivative of both sides, with respect to time t. Then we have that

$$d(t)^2 + h(t)^2 = l(t)^2$$

becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[d(t)^2 + h(t)^2\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[l(t)^2\right].$$

We now note that while the depth d and height h of the ladder away from the wall and above the ground (respectively) change, the length l of the ladder does not! Therefore d and h are functions with respect to t, while l is just a constant! (It is technically a constant *function,* but for the sake of this problem, we can treat it as a constant). Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[d(t)^2 + h(t)^2\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[l^2\right].$$

is nothing but

$$2d(t)d'(t) + 2h(t)h'(t) = 0$$

or written more compactly

$$2dd' + 2hh' = 0$$

via implicit differentiation. This is welcome news, as we now have a single formula that relates d, d', h, and h' to each other! Now all the remains is to plug in the values we know and solve for what we do not!

$$\begin{aligned} 2(8)(3) + 2(6)(h') &= 0 \\ 48 + 12h' &= 0 \\ 12h' &= -48 \\ h' &= -\tfrac{48}{12} \\ h' &= -4\,{}^{\mathrm{ft}}\!/_{\mathrm{s}}. \end{aligned}$$

Therefore the ladder is moving at $-4\,\text{ft}/\text{s}$, or $4\,\text{ft}/\text{s}$ downwards. ■

Notice in this problem we

1. Assigned functions with respect to time to each piece of information we were given. That is $l(t)$, $d(t)$, $h(t)$, $d'(t)$, and $h'(t)$.
2. Determined which of those pieces of information are constant. In our case, $l(t)$.
3. Determine the formula that relates the information. In our case, it was $d^2 + h^2 = l^2$.
4. Use that formula to find any missing information possible. In our case it was d, the depth of the ladder.
5. Take the derivative of both sides of the formula with respect to time. In our case, we obtained that $2dd' + 2hh' = 0$.
6. Use that new formula to relate all the information given. This was us plugging in the values.
7. Solve for the missing piece of information. This is how we obtained h'.

Note 3.3.1 — The Method of Solving Related Rates Problems. While there is no one-size-fits-all approach to solving related rates problem, they all do follow the same sort of logic.

1. Assign functions with respect to time to each piece of information we are given.
2. Determine which of those pieces of information are constant.
3. Determine the formula that relates the information.
4. Use that formula to find any missing information possible.
5. Take the derivative of both sides of the formula with respect to time.
6. Use that new formula to relate all the information given.
7. Solve for the missing piece of information.

Exercise 3.3 Suppose a five-foot-ladder is placed against a three-foot-tall wall. It begins slipping down along the wall at a rate of $2\,\text{ft}/\text{s}$. When the top of the ladder is two feet above the ground, at what rate is the ladder moving away from the wall? ■

3.4 Local Extrema

In our capitalistic society, one of the most important classes of problems are minimisation and maximisation problems. These problems occur when a company wants to maximize profits while minimizing cost, or when a company desires to minimize space used by their packaging, or if they wished to maximize the return-on-investment for a new machine they purchased.

For all of these things, there's Calculus. First, we'll define the absolute maximum and absolute minimum.

Definition 3.4.1 — Absolute Maximum. Let f be a function defined on a closed interval I and let $c \in I$. We say that f has an ABSOLUTE MAXIMUM on I at c if $f(c) \geq f(x)$ for all $x \in I$.

Definition 3.4.2 — Absolute Minimum. Let f be a function defined on a closed interval I and let $c \in I$. We say f has an ABSOLUTE MINIMUM on I at c if $f(c) \leq f(x)$ for all $x \in I$.

The definitions of both of these terms may look complicated, but it is easy enough to think about it this way: take some interval $I = [a,b]$. For a function f on that interval, its maximum is its largest output value, and its minimum is its smallest output value.

Note 3.4.1 Collectively, the maxima (plural of maximum) and minima (plural of minimum, blame the Romans) are called 'extrema.' When the text uses the word extrema, we mean both the minimum and maximum values the function takes on the interval I.

3.5 The Extreme Value Theorem

Theorem 3.5.1 — Extreme Value Theorem. If f is a continuous function over the closed, bounded interval $[a,b]$ there is a point in $[a,b]$ at which f has an absolute maximum over $[a,b]$ and there is a point in $[a,b]$ at which t has an absolute minimum over $[a,b]$.

Definition 3.5.1 — Local Maxima. A function f has a LOCAL MAXIMUM at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$.

Definition 3.5.2 — Local Minima. A function f has a LOCAL MINIMUM at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)\ \forall X \in I$.

Definition 3.5.3 — Inflection Point. A function f has an INFLECTION POINT at c if and only if $f''(c) = 0$ and for some $\varepsilon > 0$, $f''(c+\varepsilon)$ and $f''(c-\varepsilon)$ have different signs.

Definition 3.5.4 — Stationary Point. A stationary point c of a function f is a point at which $f'(c) = 0$. A stationary point is the general name for either minima, maxima, or inflection points.

Definition 3.5.5 — Critical Points. Let I be a closed, bounded interval with $c \in I$ and $c \in \mathbb{R}$. If $f(c)$ exists, we say c is a CRITICAL POINT of f if $f'(c) = 0$ or if $f'(c)$ does not exist.

Corollary 3.5.2 — Extreme Value Location Theorem. Let f be defined on $[a,b]$. The absolute extrema of f must occur at either the end points (a or b) or the critical points of f.

■ **Example 3.5** Find the critical points of the function $f(x) = x^3 - 4x^2 + 3x - 1$.

The definition of a critical point is a point $c \in \mathbb{R}$ such that $f'(c) = 0$ or $f'(c)$ does not exist. Then first we will find the derivative $f'(c)$, then set that derivative equal to zero and solve for c. Since $f(c) = c^3 - 4c^2 + 3c - 1$, the power rule gives us that $f'(c) = 3c^2 - 8c + 3$. Since we want $f'(c) = 0$, we have

$$3c^2 - 8c + 3 = 0.$$

Since this is just a quadratic function with respect to the variable c, we may use the quadratic formula to obtain

$$\begin{aligned} c &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(3)}}{3 * 2} \\ &= \frac{8 \pm \sqrt{64-36}}{6} \\ &= \frac{8 \pm \sqrt{28}}{6} \\ &= \frac{8 \pm 2\sqrt{7}}{6} \\ &= \frac{4 \pm \sqrt{7}}{3} \end{aligned}$$

We say the critical points of f are at $\left(\frac{4+\sqrt{7}}{3}, f\left(\frac{4+\sqrt{7}}{3}\right)\right)$ and $\left(\frac{4-\sqrt{7}}{3}, f\left(\frac{4-\sqrt{7}}{3}\right)\right)$. Evaluated, these are approximately $(0.451, -0.369)$ and $(2.215, -3.113)$, respectively. ■

Exercise 3.4 In the last example, we only found where $f'(c) = 0$, even though critical points can happen at places where $f'(c)$ does not exist! What can we say about $f'(c)$ from the last example that allows us to skip this step? ■

Note 3.5.1 When finding critical points, it is customary to use the variable c. Note how we used the quadratic formula in the last example, which also has a c in its formulation. These are clearly not the same, and do not get them confused!

■ **Example 3.6** Find the critical points of $f(x) = \sin x$ on $[0, 2\pi]$.

$$\begin{aligned} f'(c) &= 0 \\ \frac{d}{dc}[\sin c] &= 0 \\ \cos c &= 0. \end{aligned}$$

Clearly, finding where $\cos c = 0$ for $c \in [0, 2\pi]$ is as simple as looking at the unit circle. Doing so, we can see that

$$c = \left\{ \frac{\pi}{2}, \frac{3\pi}{2} \right\}.$$

■

Exercise 3.5 In the last example, we only found where $f'(c) = 0$, even though critical points can happen at places where $f'(c)$ does not exist! What can we say about $f'(c)$ from the last example that allows us to skip this step? ■

Theorem 3.5.3 — Fermat's Theorem. If f has a local extremum at c and f is differentiable at c then $f'(c) = 0$.

Exercise 3.6 What is the intuition behind Fermat's Theorem? Try drawing a picture of some function with local extremum to make sense of it. ■

■ **Example 3.7** Find the absolute extrema of $f(x) = x^3 + 3x^2 - 2x - 4$ on $[-3, 2]$.

First, we'll check the end points of the original function, since extrema can happen there. Since the end points are $x = -3$ and $x = 2$, we will find $f(-3)$ and $f(2)$:

$$\begin{aligned} f(-3) &= (-3)^3 + 3(-3)^2 - 2(-3) - 4 \\ &= -27 + 27 + 6 - 4 \\ &= 2, \end{aligned}$$

$$\begin{aligned} f(2) &= (2)^3 + 3(2)^2 - 2(2) - 4 \\ &= 8 + 12 - 4 - 4 \\ &= 12. \end{aligned}$$

Now we check for critical points. Since the critical points of f happen where its derivative is zero, we find its derivative and set it equal to zero. Note that we're using the derivative here to give us information about the original function. Since the derivative of f is $f'(x) = 3x^2 + 6x - 2$, we are finding $f'(c) = 0$ or

$$3c^2 + 6c - 2 = 0.$$

Therefore we will use the quadratic formula to obtain

$$\begin{aligned}
c &= \frac{-(6) \pm \sqrt{(6)^2 - 4(3)(-2)}}{2(3)} \\
c &= \frac{-6 \pm \sqrt{36 + 24}}{6} \\
c &= \frac{-6 \pm \sqrt{60}}{6} \\
c &= \frac{-6 \pm \sqrt{4 \cdot 15}}{6} \\
c &= \frac{-6 \pm 2\sqrt{15}}{6} \\
c &= \frac{-3 \pm \sqrt{15}}{3} \\
c &= \left\{ \frac{-3 + \sqrt{15}}{3}, \frac{-3 - \sqrt{15}}{3} \right\} \\
c &\approx \{0.29, -2.29\}.
\end{aligned}$$

This gives us two potential inputs to our original function f to test. We must find both $f\left(\frac{-3+\sqrt{15}}{3}\right)$ and $f\left(\frac{-3-\sqrt{15}}{3}\right)$. Going through the process of finding $f\left(\frac{-3+\sqrt{15}}{3}\right)$ is as simple as plugging in the value:

$$f\left(\frac{-3+\sqrt{15}}{3}\right) = \left(\frac{-3+\sqrt{15}}{3}\right)^3 + 3\left(\frac{-3+\sqrt{15}}{3}\right)^2 - 2\left(\frac{-3+\sqrt{15}}{3}\right) - 4 \approx -4.30.$$

We do the same for $f\left(\frac{-1-\sqrt{15}}{3}\right)$, plugging in the value:

$$f\left(\frac{-3-\sqrt{15}}{3}\right) = \left(\frac{-3-\sqrt{15}}{3}\right)^3 + 3\left(\frac{-3-\sqrt{15}}{3}\right)^2 - 2\left(\frac{-3-\sqrt{15}}{3}\right) - 4 \approx 4.30.$$

Therefore, we have four points from which to choose the absolute extrema. Keep in mind that the absolute minimum and absolute maximum are just the smallest and largest values that the output obtains, so we're looking over the y-values. Our list is just the two points that form the endpoints $(-3, 2)$ and $(2, 12)$ and also the two critical points we found $(0.29, -4.30)$ and $(-2.29, 4.30)$. Looking at the output y-values of each tells us that the absolute maximum output value is twelve, so the absolute maximum is the point $(2, 12)$. The absolute minimum output y-value is found similarly, and we get that the absolute minimum must be $(0.29, -4.30)$. ■

3.6 Derivative Testing

We know that continuous functions on closed intervals have their extrema either at their endpoint or at their critical points. If the function has a local extremum at a critical point, then the function's derivative changes signs there. Therefore to determine the extrema, we must test the sign of the function's derivative to the left and right of each critical point. This process is called the FIRST DERIVATIVE TEST. The process is stated more formally below.

3.6.1 The First Derivative Test

Definition 3.6.1 — First Derivative Test. Let f be continuous at a stationary point c.

1. If $f'(c) > 0$ on an open interval extending left from c and $f'(c) < 0$ on an open interval extending right from c, then f has a local (and perhaps absolute) maximum at c.
2. If $f'(c) < 0$ on an open interval extending left from c and $f'(c) > 0$ on an open interval extending right from c, then f has a local (and perhaps absolute) minimum at c.
3. If $f'(c)$ has the same sign on an open interval extending left from c and on an open interval extending right from c, then f has an inflection point at c.

■ **Example 3.8** Given $f(x) = \frac{1}{3}x^3 + 2x^2 - 3x - 2$, find its local extrema, determine also the intervals on which it is increasing and decreasing, and find any inflection points of the function, if it has them.

We begin by finding critical points, and the first step of that is noting that $f'(x) = x^2 + 4x - 3$. Setting this equal to zero and solving $f'(c) = 0$ gives

$$\begin{aligned}
c^2 + 4x - 3 &= 0 \\
c &= \tfrac{-(4)\pm\sqrt{(4)^2-4(1)(-3)}}{2(1)} \\
c &= \tfrac{-4\pm\sqrt{16+12}}{2} \\
c &= \tfrac{-4\pm\sqrt{16+12}}{2} \\
c &= \tfrac{-4\pm\sqrt{28}}{2} \\
c &= \tfrac{-4\pm2\sqrt{7}}{2} \\
c &= -2\pm\sqrt{7} \\
c &\approx \{-4.65, 0.65\}
\end{aligned}$$

Then we have our two critical points. The actual First Derivative Test is setting up a table like what is below, in which we list every critical point, leaving space in between them for arbitrary values that we choose. Remember, this test is only interested in the signs of the numbers we find, and nothing else. The setup looks like this:

Choose	Critical Point	Choose	Critical Point	Choose
	-4.65		0.65	
	0		0	

We placed the critical points themselves in. The row below them is the value of the derivative at that point, and since we found these by setting the derivative equal to zero, their derivative is zero. The row below that is left blank, as we do not yet have enough information, but it will be where we will put if the critical point in question is a minimum, maximum, or inflection point. The next step is to choose values. We must choose a point to the left of -4.65, between -4.65 and 0.65, and to the right of 0.65. The easiest numbers for those choices, in order, are -5, 0, and 1, so we will choose them. Note that we could also choose literally any numbers that meet the criteria, though. Making the sensible choices gives us the following table:

Choose	Critical Point	Choose	Critical Point	Choose
-5	-4.65	0	0.65	1
$f'(-5)$	0	$f'(0)$	0	$f'(1)$

For each choice, we must plug it into our derivative to find out the sign. We find $f'(-5) = (-5)^2 + 4(-5) - 3 = 2$, so we note that $f'(-5)$ is positive. We do the same for $f'(0)$ and $f'(1)$, noting that they are negative and positive, respectively. Then our table looks like:

CHOOSE	CRITICAL POINT	CHOOSE	CRITICAL POINT	CHOOSE
-5	-4.65	0	0.65	1
$f'(-5)$	0	$f'(0)$	0	$f'(1)$
$+$		$-$		$+$

The only thing left to do is use the information in the table to determine if the function f is increasing or decreasing at the critical points. At the critical point for $c = -4.65$, we note that the function is increasing to the left (since its derivative has a positive slope there) and decreasing to the right. The only way this can happen is if that value of c is a maximum. In a similar fashion, we know that around the critical point at $c = 0.65$, the function is decreasing to the left and increasing to the right, therefore it has a minimum there. Filling out the rest of the table gives the final version.

CHOOSE	CRITICAL POINT	CHOOSE	CRITICAL POINT	CHOOSE
-5	-4.65	0	0.65	1
$f'(-5)$	0	$f'(0)$	0	$f'(1)$
$+$	MAXIMUM	$-$	MINIMUM	$+$

The question asked us to find the minimum and maximum values, which we have done. There are no inflection points, since those would happen if the sign were the same on either side of a critical point. The only thing left to do is state where the function is increasing or decreasing. Reading the chart tells us that we are increasing on $(-\infty, -4.65) \cup (0.65, \infty)$ and decreasing on $(-4.65, 0.65)$. Note the open intervals here, since at the critical points themselves, we are neither increasing nor decreasing (we are stationary, in fact, since they're stationary points as well as critical points!)

■

3.6.2 The Second Derivative Test

Another way to determine extrema along with concavity is the SECOND DERIVATIVE TEST. This test is usually used as a shortcut to find inflection points, since the second derivative of a function at a point being zero implies that concavity changes at that point.

Definition 3.6.2 — The Second Derivative Test. Suppose f is a function that is twice differentiable at a stationary point c.

1. If $f''(c) > 0$, then f is concave up at that point.
2. If $f''(c) < 0$, then f is concave down at that point.
3. If $f''(c) = 0$ and for some $\varepsilon > 0$ we have that $f''(c+\varepsilon)$ and $f''(c-\varepsilon)$ have different signs, then c is an inflection point.

R If the Second Derivative Test is inconclusive, that only means that it doesn't tell us any information!

Note 3.6.1 Keep in mind that the Second Derivative Test is a rarely-used tool that is mentioned here only as a possible time-saver in the future. The First Derivative Test can tell all of the information that its little brother can! Also keep in mind that the function f must have two derivatives that exist, and both f' and f'' can't both be zero.

■ **Example 3.9** Consider the function $f(x) = x^3 - 6x^2 + 9x + 30$. Find its inflection points, if any.

In order to apply the Second Derivative Test, we must find the second derivative of our function f. Since the first derivative is $f'(x) = 3x^2 - 12x + 9$, we know that $f''(x) = 6x - 12$. Since we're looking for inflection point, we'll use the third part of the test. Basically, we'll set the second derivative equal to zero and solve, and then check each point we find by picking points on its left and right. If the sign changes, there is an inflection point there. Solving $6c - 12 = 0$ gives that $c = 2$, so we only need to pick numbers on either side. Picking the easiest numbers possible, we'll choose $x = 0$ for the number to the left and $x = 3$ for the number to the right, though any numbers that meet that criteria would work! The table looks like the following:

CHOOSE	INFLECTION POINT	CHOOSE
0	2	3
$f''(0)$	$f''(2)$	$f''(3)$
-12	0	$+6$

Notice how $f''(0)$ is negative and $f''(3)$ is positive? Then the function has an inflection point at $x = 2$. The exact inflection point would be $(2, f(2))$ or the point $(2, 32)$. If the sign had been different on either side of the point we tested, we would not be able to call it an inflection point! ■

3.7 The Mean Value Theorem

The Mean Value Theorem (MVT) is used to justify proofs related to the first and second derivative tests, and is related, as one would expect, to the mean value of a function. Before we get to it, we need to talk about the theorem behind the idea of critical points.

Theorem 3.7.1 — Rolle's Theorem. Suppose $f(x)$ is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists a number c such that $a < c < b$ and $f'(c) = 0$. That is, $f(x)$ has a critical point in the interval (a, b).

We can use Rolle's Theorem to prove statements about the roots of a function, but its main use is to prove the MVT, which we state now.

Theorem 3.7.2 — The Mean Value Theorem. Suppose $f(x)$ is a function that is continuous on $[a, b]$ and differentiable on (a, b). Then there exists some number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note 3.7.1 Note that we can multiply through both sides of the above equation by $b - a$ (since $b - a \neq 0$ because $a \neq b$ from the conditions). We then obtain an alternate MVT form of

$$f(b) - f(a) = f'(c)(b - a).$$

Note 3.7.2 The MVT simply states that there exists a c such that the tangent line to the function curve at $(c, f(c))$ and the secant line between $(a, f(a))$ and $b, f(b)$ must be parallel, assuming f is continuous and differentiable on the required intervals.

■ **Example 3.10** Determine all the values of c that satisfy the MVT for the function $f(x) = x^3 + x^2 - x$ on the interval $[-1, 1]$.

In order to apply the MVT, we must first note that $f(x)$ is a polynomial, and thus everywhere continuous and differentiable, particularly on $[-1, 1]$ where required. Then we find

$$f'(c) = \frac{f(1) - f(-1)}{(1) - (-1)} = \frac{\left((1)^3 + (1)^2 - (1)\right) - \left((-1)^3 + (-1)^2 - (-1)\right)}{(1) - (-1)} = \frac{(1) - (1)}{2} = 0.$$

We now find that $f'(c) = 3c^2 + 2c - 1$, so we must find

$$3c^2 + 2c - 1 = 0.$$

This is a simple application of the quadratic formula, thus we have that

$$c = \frac{-2 \pm \sqrt{(-2)^2 - 4(3)(-1)}}{2(3)} = \frac{-2 \pm \sqrt{4 + 12}}{6} = \frac{-2 \pm 4}{6} = \left\{ \frac{1}{3}, -1 \right\}.$$

Since the solution of $c = -1 = a$ and the MVT requires $a < c < b$, we exclude $c = -1$. This leaves our only solution of $c = \frac{1}{3}$. If we plot the tangent line through the point $\left(\frac{1}{3}, f\left(\frac{1}{3}\right)\right)$ together with the graph of f and the secant line between $(-1, f(-1))$ and $(1, f(1))$ on the same graph, we can see what is going on pretty clearly.

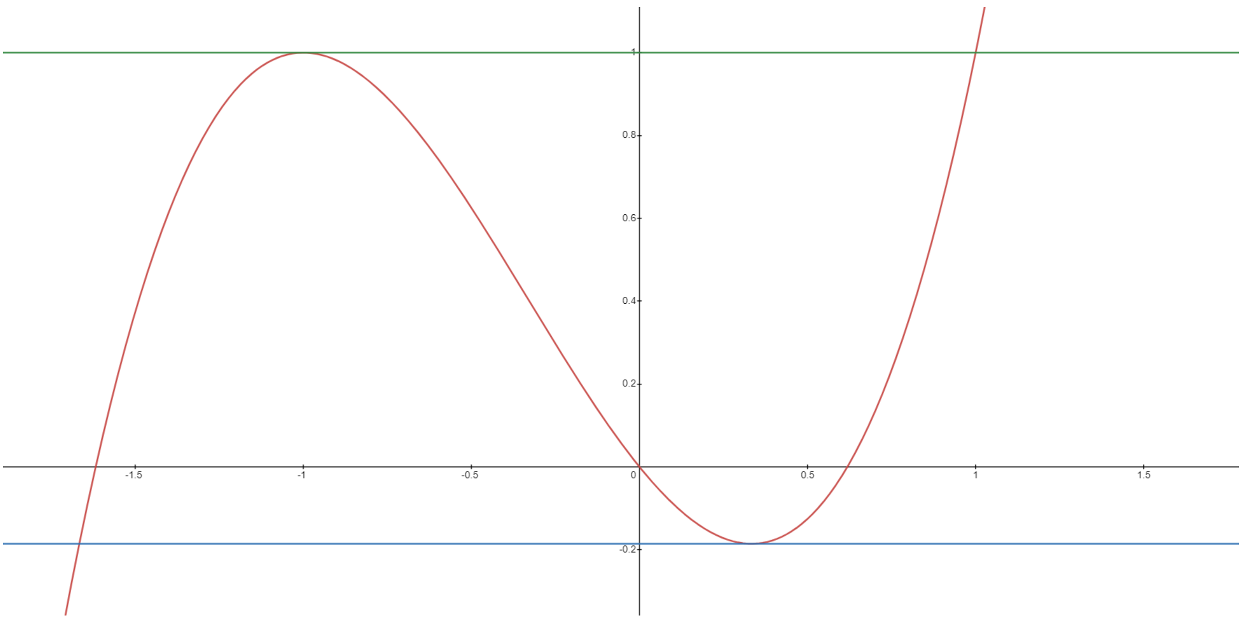

Figure 3.2: A graph of f, together with a related secant and tangent line.

We notice how the secant and tangent lines are parallel, and how there did, in fact, exist a c that made this situation happen, exactly as the MVT foretold! ■

Exercise 3.7 Determine all the values of c that satisfy the MVT for the function $f(x) = 4x^3 + 3x^2 - 2$ on the interval $[1,4]$. ■

3.8 Limits at Infinity

Previously, we've only dealt with limits going to finite values. Namely, for $a \in \mathbb{R}$ and some function that meets the criteria to be defined around a, we've talked about

$$\lim_{x \to a} f(x).$$

The natural question that arises now is what happens if we let a approach infinity? We've actually already done something with this concept when we discussed end behavior, we just used different notation. When analyzing the function $f(x) = \frac{1}{x}$, we said 'as $x \to \infty$, $f(x) \to 0$, and this was talking about the horizontal asymptote that the f has at the x-axis. In this next example, we'll consider it using limit notation instead.

3.8.1 Infinite Limits Using Tables

■ **Example 3.11** Give a value for $\lim\limits_{x \to \infty} \dfrac{1}{x}$, if it has one.

It helps to make a table in this instance. On the left-hand side we'll place the inputs, and on the right-hand side we'll put the result when we run the inputs to the function. Since $x \to \infty$, we should start with a small value of x and quickly get large, preferably jumping by orders of magnitude each time. So we'll choose $x = 1$, $x = 10$, $x = 100$, and $x = 1000$, and that should be enough to establish a pattern. If not, we can always choose more points later. The table, once it is filled out, looks like the following:

x	$\frac{1}{x}$
1	$1/1$
10	$1/10$
100	$1/100$
1000	$1/1000$

Clearly the values are getting smaller and smaller. If we consider that this pattern will continue forever, it becomes pretty clear that our function is approaching zero. In this case, it is convention to say that the limit equals zero. That is,

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

■

■ **Example 3.12** Find a value for $\lim\limits_{x \to -\infty} \dfrac{\cos x}{x}$, if it has one.

This problem is exactly the same as the last, but it may require an extra part of the table to store the raw input. Notice also that we are approaching negative infinity instead. In this case, the choices for x-values should simply be getting more negative. So we'll choose $x = -1$, $x = -10$, $x = -100$, and $x = -1000$, and that should be enough to establish a pattern.

R Note that we could also choose multiples of π here, since our function is a trigonometric one. While that would make computing values of the cosine function easier, we would still be using a calculator when we had to divide by π. We'll just stick to the basics.

x	$\frac{\cos x}{x}$	Value
-1	$\frac{\cos(-1)}{-1}$	-0.5403
-10	$\frac{\cos(-10)}{-10}$	0.0839
-100	$\frac{\cos(-100)}{-100}$	-0.0086
-1000	$\frac{\cos(-1000)}{-1000}$	-0.0006

Even though our value changes signs, notice how it is getting smaller and smaller by orders of magnitude. Then we may conclude that

$$\lim_{x\to-\infty} \frac{\cos x}{x} = 0.$$

■

As usual with mathematics, there is more than one way to solve a problem, like in the next example. We'll first use the table method without commentary, and then we'll go through using the Algebraic Limit Theorem (limit laws) to solve the same problem using a previous result.

■ **Example 3.13** Find a value for $\lim_{x\to\infty}\left(\frac{1}{x}+4.2\right)$, if it has one.

x	$\frac{1}{x}+4.2$
1	$\frac{1}{1}+4.2=5.2$
10	$\frac{1}{10}+4.2=4.3$
100	$\frac{1}{100}+4.2=4.21$
1000	$\frac{1}{1000}+4.2=4.201$
10000	$\frac{1}{10000}+4.2=4.20001$

Another method is to simply use the limit laws. That is, since we are given $\lim_{x\to\infty}\left(\frac{1}{x}+4.2\right)$, we may 'distribute' the limit to obtain

$$\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}4.2.$$

Since we know that $\lim_{x\to\infty}\frac{1}{x}=0$, and $\lim_{x\to\infty}4.2=4.2$, we may substitute those values to see that $(0)+(4.2)=4.2$, as expected.

■

3.8.2 Infinite Limits that Diverge

Sometimes, limits at infinity don't exist. This is usually because they're either defined only on a restricted domain or because they never 'settle' on a value. In this case, we say the limit at infinity DIVERGES.

■ **Example 3.14** Give a value for $\lim_{x\to\infty}\cos x$, if it has one.

While we could build a table for this function, it wouldn't be terribly useful. Keep in mind that $\cos x$ is a periodic function, and therefore only repeats as $x\to\infty$. Since it never settles on a value, the limit does not exist, and we are done. ■

■ **Example 3.15** Give a value for $\lim_{x\to\infty}(-x^2)$, if it has one.

It should be obvious from our knowledge that $f(x)=-x^2$ is a parabola with vertex at the origin that opens downward. As $x\to\infty$ (which is sometimes read "as x gets big,") it is clear that the output approaches negative infinity. In this case, while the limit doesn't exist and diverges, we do have a general sense of where it is going, so we say that

$$\lim_{x\to\infty}(-x^2) = -\infty.$$

■

3.8.3 Infinite Limits of Rational Functions

When we're talking about infinite limits of rational functions, there are three possible cases. First, recall the definition of a rational function:

Definition 3.8.1 — Rational Function. A RATIONAL FUNCTION is a function of the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are both polynomial functions and $Q(x) \neq 0$.

Definition 3.8.2 — Limit of a Rational Function. Let $\deg(P)$ be the degree of the polynomial function P. Consider the infinite limit of

$$\lim_{x\to\infty}\frac{P(x)}{Q(x)}.$$

There are three cases

1. If $\deg(P) < \deg(Q)$, then $\lim_{x\to\infty}\frac{P(x)}{Q(x)} = 0$.
2. If $\deg(P) > \deg(Q)$, then $\lim_{x\to\infty}\frac{P(x)}{Q(x)} = \infty$.
3. If $\deg(P) = \deg(Q)$, then $\lim_{x\to\infty}\frac{P(x)}{Q(x)} = \frac{p}{q}$, where p and q are the leading coefficients of $P(x)$ and $Q(x)$, respectively.

■ **Example 3.16** Determine $\lim_{x\to\infty}\frac{x^2+3x-1}{x-4}$.

Notice that $\deg(x^2+3x-1) > \deg(x-4)$. Therefore $\deg(P) > \deg(Q)$ and

$$\lim_{x\to\infty}\frac{x^2+3x-1}{x-4} = \infty.$$

■

■ **Example 3.17** Determine $\lim_{x\to-\infty}\frac{x^2-7}{2x+3x^4}$.

Notice that $\deg(x^2-7) < \deg(2x+3x^4)$. Therefore $\deg(P) < \deg(Q)$ and

$$\lim_{x\to-\infty}\frac{x^2-7}{2x+3x^4} = 0.$$

■

■ **Example 3.18** Determine $\lim_{x\to\infty}\frac{4x^3-7}{5x^2+2x^3+1}$.

Notice that $\deg(4x^3-7) = \deg(5x^2+2x^3+1)$. Therefore $\deg(P) = \deg(Q)$ and

$$\lim_{x\to\infty}\frac{4x^3-7}{5x^2+2x^3+1} = \frac{4}{2} = 2.$$

■

3.9 Strange Limits and Derivatives

We have worked with taking limits of functions at zero and limits of functions at infinity, but consider the following limit:

$$\lim_{x\to\infty} \frac{4x+2}{3x-2}$$

Sharp-minded readers know that adding and subtracting 'at infinity' doesn't actually do much, and we can therefore entirely ignore the $+2$ and -2 in the numerator and denominator, respectively. This doesn't feel like good math though, does it? Notice how if we 'plug in' ∞ to the top and bottom you obtain

$$\frac{4\cdot\infty+2}{3\cdot\infty-2} = \frac{4\cdot\infty}{3\cdot\infty} = \frac{\infty}{\infty}.$$

At this point, we may be tempted to 'cancel' the two infinities. It is said that if we cancel infinities like this, the ghost of Newton will appear in the reader's bedroom late at night and whisper horrible secrets that will haunt the reader until the end of their days. Do not cancel infinities, mathematics does not work that way. Remember that infinity is mysterious and spooky and does not behave like anything else in mathematics. In fact, the form $^{\infty}/_{\infty}$ is a construct in mathematics known as an indeterminate form. This is the strangest degree of no answer we have.

3.9.1 Indeterminate Forms

We begin with a very powerful theorem, and then we define the concept of an indeterminate form.

Theorem 3.9.1 — The Algebraic Limit Theorem. Let f and g be real- or complex-valued functions. Taking the limit of an operation $(+,-,\cdot,\text{etc})$ on f and g under particular conditions works with the limits of f and g. More specifically, see the limit laws from Chapter 1.

Definition 3.9.1 — Indeterminate Form. An INDETERMINATE FORM is an expression involving $0, 1,$ and ∞ that is obtained by applying the Algebraic Limit Theorem in the process of trying to determine the limit. This form does not restrict the limit to any specific value or a form of infinity, and doesn't actually give an specific determination of the limit in question.

Perhaps the clearest example of an indeterminate form would be as follows:

■ **Example 3.19** Let $c \in \mathbb{R}$. Consider the limit

$$\lim_{x\to 0} \frac{cx}{x} = c.$$

So this limit can converge to any value c, even though each corresponds to the indeterminate form $\frac{0}{0}$. ■

3.9.2 L'Hôpital's Rule

Theorem 3.9.2 — L'Hôpital's Rule. Suppose f and g are differentiable functions over an open interval containing a, except possibly at a. Suppose

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}.$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided $\lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ exists.

This theorem also holds if $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{0}{0}$, and note that a can be any real number, or one of $\pm\infty$.

The basic version of what this says is this - plug in your limit value and simplify. For numbers (as in $\lim_{x\to 4}$,) it is as simple as plugging in the number (or 4, in the example). For infinity (as in $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$,) just plug in the infinity and note that a number added, subtracted, multiplied, or divided with infinity changes infinity at all. Neither does raising infinity to a finite power or taking a root of infinity.

After doing this, if you get something of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then you may find the actual limit of the original problem by simply taking the derivative of both the numerator and denominator and then plugging in again.

■ **Example 3.20** Recall the limit from earlier

$$\lim_{x\to\infty}\frac{4x+2}{3x-2}.$$

Notice that 'plugging in infinity' to the top and bottom gives $\frac{\infty}{\infty}$. Then we may use L'Hôpital's Rule. Simply take the derivative of the numerator and the denominator separately:

$$\lim_{x\to\infty}\frac{4x+2}{3x-2} = \lim_{x\to\infty}\frac{[4x+2]'}{[3x-2]'} = \lim_{x\to\infty}\frac{4}{3} = \frac{4}{3}.$$

Notice that this also works with limits like $\lim_{x\to 1}\frac{3x^2-2x-1}{1-x}$. We check to see if we can use L'Hôpital's Rule by plugging in 1 to the limit:

$$\frac{3(1)^2-2(1)-1}{1-(1)} = \frac{3-2-1}{0} = \frac{0}{0}.$$

Since we obtained $\frac{0}{0}$, we may use the rule to evaluate the limit at the given point (in this case, $a=1$). We take the derivative of the numerator and denominator and then evaluate.

Note 3.9.1 Please note that we take the derivatives of each level of the fraction separately and do not use a Quotient Rule.

$$\lim_{x\to 1}\frac{3x^2-2x-1}{1-x} = \lim_{x\to 1}\frac{[3x^2-2x-1]'}{[1-x]'} = \lim_{x\to 1}\frac{6x-2}{-1}.$$

Since 1 is okay to be plugged into the limit at this point, we do so to obtain

$$\lim_{x\to 1}\frac{6x-2}{-1} = \frac{6(1)-2}{-1} = \frac{4}{-1} = -4.$$

■

3.10 Graphing Using Calculus

So far, we have learned a few contextual applications and several theoretical uses for the mathematics we have discussed. In this final section of the chapter, we will put together the theory to sketch a graph of a function by hand. While many technology aids exist to do this exact thing, it is always good to have a solid understanding of what it is your tools are doing. We will begin with a step-by-step process that uses the tools we've discovered thusfar to find enough information about a given function to sketch it.

> **Note 3.10.1** It is important to keep in mind that not all of the following steps need be taken with every function, nor do they need to be taken in this particular order. They are simply presented in the order that is typically considered easiest to hardest to perform.

3.10.1 A Strategy for Graphing by Hand

Given some function $f(x)$, to graph by hand, do the following:

1. Check the domain, symmetry, and periodicity. They will all help with the general shape later on.
2. Find the x-intercept(s) by solving $f(x) = 0$.
3. Find the y-intercept by finding $f(0)$.
4. Find the asymptotes of $f(x)$. Horizontal asymptotes happen if the limit as $x \to \pm\infty$ equals some number. Vertical asymptotes happen at a point a if the limit as $x \to a$ equals either $\pm\infty$. Slant asymptotes occur only in rational functions where the degree of the numerator is one larger than the degree of the denominator, and are found using polynomial division.
5. Find $f'(x)$ and $f''(x)$. Use these to find the critical points where $f'(x) = 0$ or is undefined. Find possible inflection points where $f''(x) = 0$ or is undefined.
6. Use all the various tests we have discussed to find where $f(x)$ is increasing, decreasing, concave up, and concave down. Locate relative and absolute extrema and identify inflection points.
7. Use all of this information to sketch the graph.

■ **Example 3.21** Graph by hand $f(x) = 6x^3 + 2x^2 - x$. In order to accomplish this, we will proceed through our strategy step-by-step until we arrive at an answer.

1. We know the domain of any polynomial is all real numbers, and while there maybe symmetry here, it isn't obvious at a glance. We know that polynomials are not periodic, so we proceed to the next step.
2. We solve $f(x) = 6x^3 + 2x^2 - x = 0$ to find the x-intercepts. We first factor x from each term to obtain $x(6x^2 + 2x - 1) = 0$ and note that implies that either $x = 0$ or $6x^2 + 2x - 1 = 0$ by the Zero Factor Property. We use the quadratic formula with $a = 6$, $b = 2$, and $c = -1$ on the remaining degree two polynomial to obtain that
$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(6)(-1)}}{2(6)} = \frac{-2 \pm \sqrt{28}}{12} = \frac{-2 \pm 2\sqrt{7}}{12} = \frac{-1 \pm \sqrt{7}}{6}.$$
This gives us intercepts at approximately $(-0.61, 0)$ $(0.27, 0)$ and .
3. We next find the y-intercept by solving $f(0) = 6(0)^3 + 2(0)^2 - (0)$, which clearly puts the intercept at the origin $(0, 0)$.
4. Since we know that the leading term $6x^3 \to \pm\infty$ as $x \to \pm\infty$, we know this polynomial has no horizontal asymptotes and has the same end behavior as $f(x) = x^3$. We know that polynomials are continuous on their domain and therefore have no vertical asymptotes.
5. We find $f'(x) = 18x^2 + 4x - 1$ and check $f'(x) = 18x^2 + 4x - 1 = 0$ using the quadratic formula to discover critical points at
$$x = \frac{-2 \pm \sqrt{22}}{18} \approx \{-0.37, 0.15\},$$

which implies the critical points are on the original graph of $f(x)$ at $(x_1, f(x_1))$ and $(x_2, f(x_2))$, or $(-0.37, f(-0.37))$ and $(0.15, f(0.15))$. We find

$$f(-0.37) = 6(-0.37)^3 + 2(-0.37)^2 - (-0.37) \approx 0.34,$$

and then we find

$$f(0.15) = 6(0.15)^3 + 2(0.15)^2 - (0.15) \approx -0.08.$$

We therefore have critical points at $(-0.37, 0.34)$ and $(0.15, -0.08)$. We then find $f''(x) = 36x + 4$, and check $f''(x) = 0$ to find the x- coordinate of the inflection point, namely

$$x = -\frac{4}{36} = -\frac{1}{9}.$$

We then find

$$f\left(-\frac{1}{9}\right) = 6\left(-\frac{1}{9}\right)^3 + 2\left(-\frac{1}{9}\right)^2 - \left(-\frac{1}{9}\right) = -\frac{19}{243} \approx -0.08.$$

Which implies there is an inflection point at $\left(\frac{1}{9}, -0.08\right)$.

6. We now place our critical points and choose values to determine increasing and decreasing. We need to find the sign of $f\prime(-1)$, which is positive, $f\prime(0)$, which is negative, and $f\prime(1)$, which is positive. That therefore makes the critical point at $(-0.37, 0.34)$ a maximum and the critical point at $(0.15, -0.08)$ a minimum.

Choose	Critical Point	Choose	Critical Point	Choose
-1	-0.37	0	0.15	1
$f'(-1)$	0	$f'(0)$	0	$f'(1)$
$+$	Maximum	$-$	Minimum	$+$

7. We leave the sketching to the reader as an exercise.

■

Exercise 3.8 Graph x-intercepts at $(-0.61, 0)$ and $(0.27, 0)$. Graph the y- intercept at the origin. Ensure the ends of the graph behave like $f(x) = x^3$. Graph the critical points $(-0.37, 0.34)$ and $(0.15, -0.08)$. Graph the inflection point $\left(\frac{1}{9}, -0.08\right)$. Keep in mind that $(-0.37, 0.34)$ is a maximum and $(0.15, -0.08)$ is a minimum, and sketch the only curve that meets all these criteria. Then graph $f(x) = 6x^3 + 2x^2 - x$ on the computer and compare. ■

Exercise 3.9 Graph by hand $f(x) = \frac{1}{4}x^2(x^2 - 2)$. ■

3.11 Optimization

We've previously discussed ways of finding both relative and absolute minima and maxima for a function. As it turns out, minimizing and maximizing are two extremely-important concepts routinely used by organizations that wish to, for example, minimize waste or maximize profits. We will dig into a few examples to finish out this chapter, but they all boil down to finding a formula that describes the given situation, then simply finding the extrema asked for using the tests we've already developed.

■ **Example 3.22** We are asked to enclose a rectangular yard with a fence. We have 300 feet of fencing, and the fence adjoins the house on one side, so one side does not need fenced in. What are the dimensions of the yard that will enclose the largest area?

We let the long side of our rectangle be x and note that the long side runs parallel to our building (so that we only have one x). We let the remaining two sides of our fence be length y each. We therefore have two formulas to work with. We know that the area (that we are trying to maximize) is given by

$$A = xy,$$

and we know that we are constrained (or limited) by the equation

$$300 = x + 2y,$$

which simply states that the perimeter of our three-sided fence must add to 300 feet of fencing.

Note 3.11.1 Note that if we were not building the fence against a house, we'd need to consider all four sides, so our constraint would be $300 = 2x + 2y$ instead.

Let's solve the constraining equation for x and plug it into our area function, since the question asked us to maximize area. We get that $x = 300 - 2y$, which when plugged into $A = xy$ gives that

$$A(y) = (300 - 2y)y = 300y - 2y^2.$$

We have been asked to find the largest (maximum) value on the interval $[0, 300]$. When working with theoretical numbers, we always tested the endpoints, but notice that if $y = 0$, then we basically have 300 feet of fencing directly against the house. In the same way if $y = 300$ we've just built one length of fence 300 feet perpendicular to the house, which hardly makes for a good area for a dog to run and play in!

Obviously our answer lies somewhere in the middle. We will find $A'(y) = 0$ and plug that back into $A(y)$ to find our maximum area. We know that $A'(y) = 300 - 4y$, and solving $300 - 4y = 0$ gives that $y = 125$ feet. This makes our optimal area

$$A(125) = 300(125) - 2(125)^2 = 6250\text{ft}.$$

Note that the question asked for the dimensions, not the area, and we've only found the y dimension so far. Simply plugging $y = 125$ back into our constraint, we obtain that

$$x = 300 - 2(125) = 300 - 250 = 50,$$

so that our dimensions are 50×250 feet. ■

Exercise 3.10 If you have 500 feet of fencing and are asked to enclose the maximum rectangular area of a field, what are the dimensions of the fence? What is its total area? ■

■ **Example 3.23** At a window construction company, a new window has been designed that is the shape of a rectangle on the bottom with a semicircular top. If we have 10 total yards of framing materials, what dimensions of this window let in the most light?

We note first that the shape that has the most area lets in the most light. Let the radius of the semicircle on top be r and the rectangle's height be h. Then the other length of the rectangle must be $2r$.

What we want to maximize is the area, which is the area of the rectangle plus the area of the semicircle. Since we know the rectangle measures $h \times 2r$, then its area is $2hr$. We know the area of semicircle is half the area of a circle or $\frac{1}{2}\pi r^2$. Then the formula we are trying to maximize is simply

$$A = 2hr + \frac{1}{2}\pi r^2.$$

Our constraint in this problem is simply the distance around (or the perimeter of) the figure. We travel distance h along the rectangle to the semicircle at the top, then distance πr along the semicircle (since it's just half of $2\pi r$, the perimeter of a full circle). Finally, we travel h again down the rectangle and $2r$ back to the start. Since this total must sum to the 10 yards of framing material we have, we have that

$$10 = 2h + 2r + \pi r.$$

Since we want to maximize area, we need to solve for a variable in the constraint equation and plug into area. Let's solve for h in the constraint and plug into our area equation. We find that

$$h = 5 - r - \frac{1}{2}\pi r.$$

Plugging that into the area equation gives that

$$A(r) = 2\left(5 - r - \frac{1}{2}\pi r\right) r + \frac{1}{2}\pi r^2 = 10r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2 = 10r - 2r^2 - \frac{1}{2}\pi r^2.$$

We then find $A'(r) = 10 - 4r - \pi r$ and $A''(r) = -4 - \pi$. Since the second derivative is always negative, the maximum area must occur at the critical point $A'(r) = 0$, which happens at

$$r = \frac{10}{\pi + 4} \approx 1.400.$$

Then we only need h, which is

$$h = 5 - (1.4) - \frac{1}{2}\pi(1.4) \approx 1.4.$$

Then the dimensions (in yards) of the rectangular window are $2r \times h$, or 2.8×1.4, and the semicircle has radius $r = 1.4$ yards. ■

■ **Example 3.24** At a window construction company, a new window has been designed that is the shape of a rectangle on the bottom with a semicircular top. If we have 14 total yards of framing materials, what dimensions of this window let in the most light? ■

III Integration

4. Introduction to Integration

4.1 Integrals as Areas

As we did with limits and derivatives, we will now investigate the notion behind the concept of an integral before we formally define it. As always, we need some notation first.

Definition 4.1.1 — The Definite Integral as Area. The integral

$$\int_a^b f(x)\mathrm{d}x$$

gives the area under the curve representing the function $f(x)$ on the interval from $[a,b]$. This is called a definite integral.

Note 4.1.1 This is read 'the integral from a to b of $f(x)$ with respect to x is the antiderivative F evaluated at b minus the antiderivative F evaluated at a.'

Note 4.1.2 The $\int$ symbol is called the 'integral symbol' and is an elongated letter 'S.' This is likely because the actual definition of that symbol is the limit of a sum, which starts with the letter 'S.'

This notation is enormously powerful, and is traditionally the subject of most of the second part of this course. It tells the area formed between the curve and the x-axis between the endpoints a and b. Area above the x-axis is positive, and area below the x-axis is negative. This is what we mean when we say the area 'under' the curve - between the curve and the x-axis. We will only be using graphs in this section, as it turns out that finding the integrals of functions is extraordinarily difficult!

■ **Example 4.1** In Figure 4.1, what is the value of the integral $\int_{-7}^{0} f(x)\mathrm{d}x$?

Notice first the bounds of our function only ask us to consider the interval from $x=-7$ to $x=0$. Then the triangle that exists on $[0,6]$ is immaterial for this problem. Since integrals discuss area, we are actually speaking of the area inside the right-angle triangle above the x-axis and between

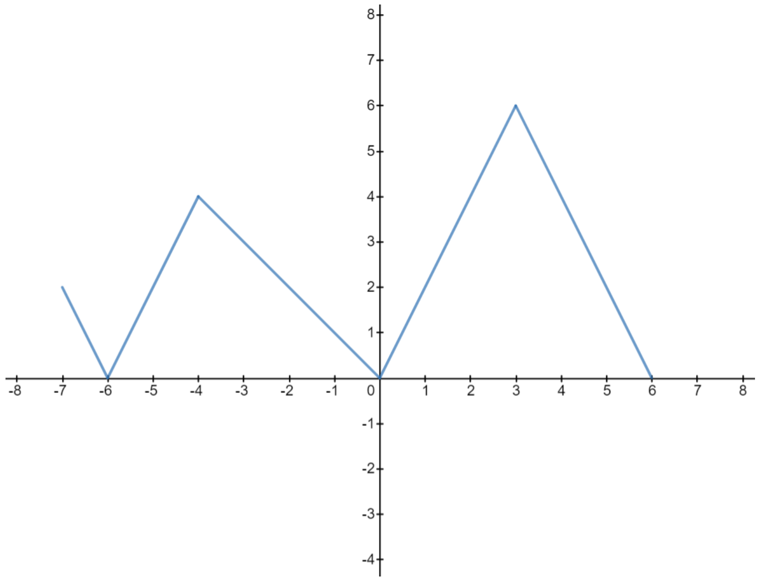

Figure 4.1: A graph to consider.

$x = -7$ and $x = -6$ and the triangle above the x-axis and between $x = -6$ and $x = 0$.

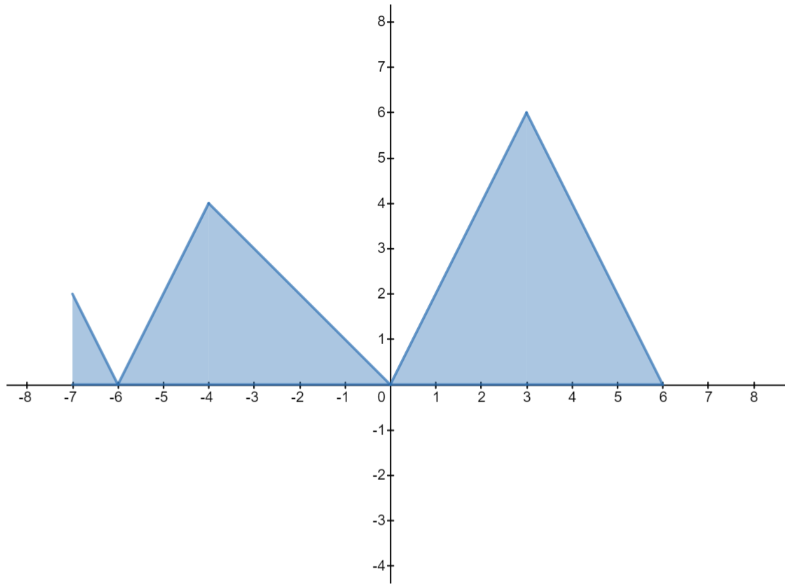

Figure 4.2: The shaded version of the same graph.

Note 4.1.3 Note that we do not care about the triangle over the interval $[0,6]$, since it is not included in the interval $[-7,0]$ that we care about.

Since there are two different triangles on the interval $[-7,0]$, we must break them up and deal with each carefully. The first small right triangle is on $[-7,-6]$ and the second is on $[-6,-0]$. We break it up like so:

$$\int_{-7}^{0} f(x)\mathrm{d}x = \int_{-7}^{-6} f(x)\mathrm{d}x + \int_{-6}^{0} f(x)\mathrm{d}x.$$

We can find the area of each triangle using the standard formula $A = \frac{1}{2}bh$. So the area of the small, left-most right triangle with base $-6-(-7) = 1$ and height 2 is given by:

$$A_1 \int_{-7}^{-6} f(x)\mathrm{d}x = \frac{1}{2}(1)(2) = 1.$$

The area of the second triangle with base $0-(-6) = 6$ and height 4 is given by

$$A_2 \int_{-6}^{0} f(x)\mathrm{d}x = \frac{1}{2}(6)(4) = 12.$$

Then the total area is simply $A_1 + A_2 = 1 + 12 = 13$.

■

Exercise 4.1 In Figure 4.2, what is the value of the integral $\int_{-6}^{6} f(x)\mathrm{d}x$? ■

4.2 Estimating Areas Using Sums

So far, we have only worked with simple antiderivatives and integrals whose graphs form basic shapes that make computation into a geometry problem instead of one for Calculus. But what if we wanted to find the area under a continuous function like $f(x) = \frac{x^2}{2}$ on the interval $I = [0,3]$?

We will do so in the next example, but first it would make sense to get an intuition behind what it is we're doing. We will take the function on the interval given and divide it up into a number of divisions called PARTITIONS.

Definition 4.2.1 — Partition of an Interval. A PARTITION of an interval $[a,b]$ on the real number line is a finite sequence $x_0, x_1, x_2, \ldots, x_n$ with each $x_n \in \mathbb{R}$ such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

After we divide the function up into partitions, we find the value at the left-hand side of each partition (which will be two different approximations). This process will form a rectangle once we connect everything up, and we know how to find the area of a rectangle! Note that the number of partitions is up to you, but we will be using four partitions for ease of computation in the example to follow.

■ **Example 4.2** Given the function $f(x) = \frac{x^2}{2}$, the closed interval of $I = [0,3]$, and the number of equal partitions, $n = 4$, solve for the left-handed approximation L_4.

Before we begin, we must first find the length of each of our four partitions. We call this value Δx (read 'delta x,') and it refers only to the length of each partition. Since the interval in general is $I = [a,b]$, the total length of the entire interval is $b-a$. Since we divided it into n equal partitions, we will have that $\Delta x = \frac{b-a}{n}$. This is always the case. For our specific example

$$\Delta x = \frac{3-0}{4} = \frac{3}{4}.$$

We begin at zero and 'hop' by $3/4$ until we reach three, which we note is just the fraction $12/4$. So we begin at 0, adding $3/4$ to get $a = x_0 = 3/4$. We then add $3/4$ to x_0 to get $x_1 = 3/4 + 3/4 = 6/4$. We then add $3/4$ again to x_1 to get $x_2 = 6/4 + 3/4 = 9/4$. Finally, we add $3/4$ again to x_2 to get $x_3 = 9/4 + 3/4 = 12/4 = 3 = b$.

We'll begin with the left-endpoint approximation using four partitions, which we label L_4. Since each rectangle formed has length Δx and height $f(x_{k-1})$, we will use the formula

$$L_4 = \sum_{k=1}^{4} f(x_{k-1})\Delta x.$$

Notice that it is x_{k-1}. This is the left-endpoint approximation. We can also factor out the Δx (since it is just a constant) and rewrite to obtain

$$L_4 = \Delta x \sum_{k=1}^{4} f(x_{k-1}).$$

Following the rules of summation, we must write out each term of our beginning at $k = 1$ and going to $k = 4$:

$$L_4 = \Delta x(f(x_{1-1}) + f(x_{2-1}) + f(x_{3-1}) + f(x_{4-1}),$$

Which is obviously just

$$L_4 = \Delta x(f(x_0) + f(x_1) + f(x_2) + f(x_3).$$

Next, we plug in the numbers that we found earlier for each x_{k-1} value. It will look like

$$L_4 = \frac{3}{4}\left(f(0) + f\left(\frac{3}{4}\right) + f\left(\frac{6}{4}\right) + f\left(\frac{9}{4}\right)\right).$$

We can recall that $f(x) = \frac{x^2}{2}$. We can plug the numbers in to get that

$$L_4 = \frac{3}{4}\left(\left(\frac{0^2}{2}\right) + \left(\frac{(3/4)^2}{2}\right) + \left(\frac{(6/4)^2}{2}\right) + \left(\frac{(9/4)^2}{2}\right)\right).$$

Squaring the numerators yields

$$L_4 = \frac{3}{4}\left(\left(\frac{0}{2}\right) + \left(\frac{(9/16)}{2}\right) + \left(\frac{(36/16)}{2}\right) + \left(\frac{(81/16)}{2}\right)\right).$$

Next, we recall that division of a fraction by a fraction is multiplications by the denominator's reciprocal, and each of our denominators is the fraction $2/1$. Then we have that

$$L_4 = \frac{3}{4}\left(\left(0 \cdot \frac{1}{2}\right) + \left(\frac{9}{16} \cdot \frac{1}{2}\right) + \left(\frac{36}{16} \cdot \frac{1}{2}\right) + \left(\frac{81}{16} \cdot \frac{1}{2}\right)\right).$$

Multiplying out each of the values and adding gives that

$$L_4 = \frac{3}{4}\left(\frac{126}{32}\right).$$

Then we simply multiply and simplify the fraction to obtain our approximation

$$L_4 = \frac{189}{64} = 2.953125.$$

That is the left-endpoint approximation. ■

Exercise 4.2 To find the right-endpoint approximation for the problem from the last example, we must do the same steps as the left-endpoint approximation, but we instead use the formula

$$R_4 = \sum_{k=1}^{4} f(x_k)\Delta x.$$

Find R_4 and compare it to L_4. Looking at a picture of what each approximation does compared to a graph of f, tell which is an under-approximation. ■

Exercise 4.3 Using the same function as the last example on the same interval, find R_6, or the right-handed approximation using six equal partitions. Knowing that the 'true value' of the area under f on our interval is $A = 4.5$, how do R_4 and R_6 compare to A, the actual area? ■

Before we go any further, we will take a quick detour and note a few properties of summations, which we will be tremendously useful in the very near future!

4.3 Properties of Sums

There are two properties that we need to know. The first says that we can factor a constant out of a sum (which is a fact we used in the last section,) and the second says that we can break up a summation across a sum or difference.

Definition 4.3.1 — Properties of Summations. If $c \in \mathbb{R}$,

1. $\sum_{k=0}^{n} ca_k = c\sum_{k=0}^{n} a_k$.
2. $\sum_{k=0}^{n} (a_k \pm b_k) = \sum_{k=0}^{n} a_k \pm \sum_{k=0}^{n} b_k$.

R Even though we begin our sums at $k = 0$ in the last definition, they can begin at any point not greater than n and still hold true!

More useful are the following four formulas for summations, each of which are proveable, though the proofs (however beautiful) are beyond the scope of the text and omitted.

Theorem 4.3.1 — Formulas for Summations. For $c \in \mathbb{R}$,

1. $\sum_{k=1}^{n} c = cn$,
2. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$,
3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$,
4. $\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4}$.

Exercise 4.4 Use the formulas and properties above to find the exact value of

$$\sum_{k=1}^{25} 5.$$

■

Exercise 4.5 Use the formulas and properties above to find the exact value of

$$\sum_{k=1}^{50} (1-k)^2.$$

■

Exercise 4.6 Use the formulas and properties above to find the exact value of

$$\sum_{k=1}^{20} k(k^2+1).$$

■

4.4 Definition of the Integral

In the last section, we talked about how to approximate using a finite number of partitions. We also saw that as we increased the number of partitions, the value approached the actual area under the curve. Using the idea of limits, which should not be a surprise at this point, we can formalize the idea of taking 'infinite' partitions. That is

Definition 4.4.1 — The Definite Integral. Given a continuous function $f(x)$ on the interval $I = [a,b]$ with n partitions each of width Δx, we choose from each interval a point x_k^*. Then the DEFINITE INTEGRAL of $f(x)$ from a to b is

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Note 4.4.1 It's worth mentioning at this point that if a function $f(x)$ is continuous on the closed interval $[a,b]$, then the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

exists and is unique. This takes us naturally to the theorem that follows.

Theorem 4.4.1 — Continuous Functions are Integrable. If $f(x)$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

It's worth mentioning that some discontinuous functions on $[a,b]$ are still integrable, as long as they're not too 'jacked up.' We'll return to this later, for now we proceed with an example of using the definite integral.

■ **Example 4.3** Given the function $f(x) = \frac{x^3}{3}$ and the closed interval $I = [1,2]$, use the definition of the definite integral to find

$$\int_1^2 f(x)\mathrm{d}x.$$

To begin this problem, we want to find our Δx, which we know to be a key part of solving the integral. As before, we do this by subtracting a from b and dividing that quantity by n, our number of partitions. However, in this problem our n is approaching infinity and therefore getting infinitely larger, so we will leave it as n in the problem. Therefore, our Δx is

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}.$$

We now want to find our x_k^* and $f(x_k^*)$, which we need to integrate our function. The * simply means that we do not care if we are looking at the right or left-hand side of our "rectangles," since in our limit 'at infinity,' each rectangle has a width of zero (and thus the same left- and right-handed endpoint). We can find our x_k^* by adding $k\Delta x$ to our lower bound of $a = 1$, giving us

$$x_k^* = 1 + k\left(\frac{1}{n}\right) = 1 + \frac{k}{n}.$$

Next, we will find our $f(x_k^*)$ by plugging our x_k^* into our function $f(x) = \frac{x^3}{3}$, and simplifying. This gives us that

$$f(x_k^*) = \frac{(1 + {}^{k}\!/\!{}_{n})^3}{3}.$$

Now, we will expand our problem and foil it our, giving us

$$f(x_k^*) = \frac{(1 + k/n)(1 + k/n)(1 + k/n)}{3} = \frac{(1 + k/n + k/n + k^2/n^2)(1 + k/n)}{3}.$$

Expanding the numerator and gathering like terms gives that

$$f(x_k^*) = \frac{1 + k/n + k/n + k^2/n^2 + k/n + k^2/n^2 + k^2/n^2 + k^3/n^3}{3} = \frac{1 + 3k/n + 3k^2/n^2 + k^3/n^3}{3}.$$

Lastly, we will get rid of the denominator by dividing each of the terms of our numerator by three. This makes our final $f(x_k^*)$ value out to be

$$f(x_k^*) = \left(1 + \frac{3k}{n} + \frac{3k^2}{n^2} + \frac{k^3}{n^3}\right)\left(\frac{1}{3}\right) = \frac{1}{3} + \frac{3k}{3n} + \frac{3k^2}{3n^2} + \frac{k^3}{3n^3}.$$

Since we found our $f(x_k^*)$, we can now integrate our function. We will start by rewriting the problem using the definition, which we restate here for ease of reading:

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Since we know that $f(x_k^*) = \left(1 + 3k/n + 3k^2/n^2 + k^3/n^3\right)(1/3) = 1/3 + 3k/3n + \dfrac{3k^2}{3n^2} + k^3/3n^3$ and $\Delta x = 1/n$, we plug those in to obtain that

$$\int_1^2 \frac{x^3}{3}\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(\frac{1}{3} + \frac{3k}{3n} + \frac{3k^2}{n^2} + \frac{k^3}{3n^3}\right)\left(\frac{1}{n}\right)\right].$$

Next, we multiply our $\frac{1}{n}$ in, giving us

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{3n} + \frac{3k}{3n^2} + \frac{3k^2}{3n^3} + \frac{k^3}{3n^4}\right).$$

We know that the sum of sums can be broken apart into individual components. Breaking up our problem gives us

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{3n} + \sum_{k=1}^{n} \frac{3k}{3n^2} + \sum_{k=1}^{n} \frac{3k^2}{3n^3} + \sum_{k=1}^{n} \frac{k^3}{3n^4}\right).$$

We now want to pull constants out of our summations. Because k is the index for the sums, we can pull n out and treat it like a constant for now. Doing this makes our problem read

$$\lim_{n\to\infty}\left(\frac{1}{3n}\sum_{k=1}^{n}1+\frac{3}{3n^2}\sum_{k=1}^{n}k+\frac{3}{3n^3}\sum_{k=1}^{n}k^2+\frac{1}{3n^4}\sum_{k=1}^{n}k^3\right).$$

With our new formulas, we can now rewrite our problem in a way that enables us to solve it. Our summations only have constants or variations of k in them, so we can restate our problem as

$$\lim_{n\to\infty}\left(\frac{1}{3n}(n)+\frac{3}{3n^2}\left(\frac{n^2+n}{2}\right)+\frac{3}{3n^3}\left(\frac{2n^3+3n^2+n}{6}\right)+\frac{1}{3n^4}\left(\frac{n^4+2n^3+n^2}{12n^4}\right)\right).$$

Then we multiply our terms together, giving us

$$\lim_{n\to\infty}\left(\frac{n}{3n}+\frac{3n^2}{6n^2}+\frac{6n^3+9n^2+3n}{18n^3}+\frac{n^4+2n^3+n^2}{12n^4}\right).$$

We now want to break our problem into individual terms to make cancellations easier. We have

$$\lim_{n\to\infty}\left(\frac{n}{3n}+\frac{3n^2}{6n^2}+\frac{3n}{6n^2}+\frac{6n^3}{18n^3}+\frac{9n^2}{18n^3}+\frac{3n}{18n^3}+\frac{n^4}{12n^4}+\frac{2n^3}{12n^4}+\frac{n^2}{12n^4}\right).$$

Next, we want to simplify our fractions, especially our n terms, to remove n from the numerators, giving us

$$\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{2}+\frac{1}{2n}+\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}+\frac{1}{12}+\frac{1}{6n}+\frac{1}{12n^2}\right).$$

We know that as $n\to\infty$, it gets infinitely larger. Because we only have n in the denominators of our fractions, those terms will get larger and larger denominators. Therefore, these terms become zero, making our problem read

$$\frac{1}{3}+\frac{1}{2}+0+\frac{1}{3}+0+0+\frac{1}{12}+0+0=\frac{1}{3}+\frac{1}{2}+\frac{1}{3}+\frac{1}{12}=\frac{4}{12}+\frac{6}{12}+\frac{4}{12}+\frac{1}{12}=\frac{15}{12}=\frac{5}{4}.$$

Then we conclude by saying that

$$\int_1^2\frac{x^3}{3}\mathrm{d}x=\frac{5}{4}.$$

■

Exercise 4.7 Use the method shown in the last example to find

$$\int_1^3 x^2\mathrm{d}x.$$

■

Exercise 4.8 Use the method shown in the last example to find

$$\int_{-2}^2 x^3\mathrm{d}x.$$

■

4.5 Antiderivatives

Before we get started with the concept of an antiderivative, let us recall that

$$\frac{d}{\mathrm{d}x}[ax^n] = anx^{n-1}.$$

This is, of course, the Power Rule from way back at the beginning of the second chapter. Not that it's at all needed, but let us recall further how to use it.

■ **Example 4.4** Find $\frac{\mathrm{d}}{\mathrm{d}x}2x^3$.

$$\frac{d}{\mathrm{d}x}2x^3 = 2\cdot 3x^{3-1} = 6x^2.$$

■

That's all well and good, but what if we were given the function $f(x) = 6x^2$ and asked 'What function did we take the derivative of to obtain $6x^2$?' The answer should be pretty straightforward to find, since we can just reverse-engineer the Power Rule. Let's first state what we must do to invoke the Power Rule.

Note 4.5.1 — Steps in the Power Rule. Given $a, n \in \mathbb{R}$, and $f(x) = ax^n$, the Power Rule states that we must

1. Multiply $f(x)$ by n, then
2. Subtract one from the exponent.

From this we obtain $f'(x) = anx^{n-1}$.

In order to 'reverse engineer' the Power Rule, we will do the opposite operations in the opposite order, that is...

Note 4.5.2 — The First Steps in the Basic Antiderivative. Given $a, n \in \mathbb{R}$, and $f(x) = ax^n$, the Basic Antiderivative states that we must

1. Add one to the exponent, then
2. Divide $f(x)$ by $n+1$.

We're almost there, but recall now how the derivative of $f(x) = ax^n$ is the same as the derivative of $g(x) = ax^n + C$, where C is a constant. Since constants are lost with derivatives, antiderivatives preserve that fact by adding that constant back, even though the exact constant is lost.

Definition 4.5.1 — The Basic Antiderivative. Given $a, n \in \mathbb{R}$ and the function $f(x) = ax^n$, the antiderivative is given by

$$F(x) = \frac{ax^{n+1}}{n+1} + C,$$

where C is an arbitrary constant term called the CONSTANT OF INTEGRATION.

Note 4.5.3 — The Steps in the Basic Antiderivative. Given $a, n \in \mathbb{R}$, and $f(x) = ax^n$, the Basic Antiderivative states that we must

1. Add one to the exponent, then
2. Divide $f(x)$ by $n+1$.
3. Add the constant of integration C.

Note 4.5.4 The antiderivative F is also called an (indefinite) integral of the function f. The notation for the (indefinite) integral of f with the respect to the variable x is notated by

$$\int f(x)\mathrm{d}x.$$

■ **Example 4.5** Find $\int 6x^2\mathrm{d}x$.

We will simply follow our process, first adding one to the exponent, and then dividing by that new exponent.

$$\int 6x^2\mathrm{d}x = \frac{6x^{2+1}}{2+1} + C = \frac{6x^3}{3} + C = 2x^3 + C.$$

■

Exercise 4.9 Find $\int 4x^7\mathrm{d}x$ ■

This 'technique' for antiderivatives works for any exponent $n \neq 1$. We'll learn what $\int x^{-1}\mathrm{d}x$ is.

■ **Example 4.6** Find $\int 6x^{-2}\mathrm{d}x$.

We will simply follow our process, first adding one to the exponent, and then dividing by that new exponent.

$$\int 6x^2\mathrm{d}x = \frac{6x^{-2+1}}{-2+1} + C = \frac{6x^{-1}}{-1} + C = -6x^{-1} + C = \frac{-6}{x} + C.$$

■

Exercise 4.10 Find $\int 4x^{-3}\mathrm{d}x$ ■

We can also take the antiderivative of a polynomial by simply taking the antiderivative of each term. First, a basic trick we'll use to help integrate constants..

Note 4.5.5 Recall that for all $a \in \mathbb{R}$ and $x \neq 0$,

$$a = a \cdot x^0.$$

Therefore

$$\int a\mathrm{d}x = \int a \cdot x^0\mathrm{d}x = \frac{a \cdot x^{0+1}}{0+1} + C = a \cdot x + C.$$

By the same logic,

$$\int 1\mathrm{d}x = x + C.$$

■ **Example 4.7** Find $\int x^2 + 2x + 1\mathrm{d}x$.

We will learn later that antiderivatives (or integrals) are just limits of sums, and both limits and sums behave (for the sorts of functions we'll see in this section, anyway) just like limits. So the integral of a sum is the sum of the integrals, and we only need apply our technique to each part. In addition, recall that $1 = 1 \cdot x^0$.

$$
\begin{aligned}
\int x^2+2x+1\mathrm{d}x &= \int x^2\mathrm{d}x+\int 2x\mathrm{d}x+\int 1\mathrm{d}x \\
&= \left(\tfrac{x^{2+1}}{2+1}+C\right)+\left(\tfrac{2x^{1+1}}{1+1}+C\right)+\int 1\cdot x^0\mathrm{d}x \\
&= \left(\tfrac{x^3}{3}+C\right)+\left(\tfrac{2x^2}{2}+C\right)+\left(\tfrac{1x^{0+1}}{0+1}+C\right) \\
&= \left(\tfrac{1}{3}x^3+C\right)+\left(x^2+C\right)+\left(\tfrac{x^1}{1}+C\right) \\
&= \left(\tfrac{1}{3}x^3+C\right)+\left(x^2+C\right)+(x+C)
\end{aligned}
$$

Here we notice that adding together three arbitrary constants just gives another arbitrary constant. That is $C+C+C=C$, since the value of C is and will always be unknown. You can write $3C$ if you like, but multiplying an unknowable constant by three just gives another unknowable constant, so convention is to ignore that fact. Then our result is simply that

$$\int x^2+2x+1\mathrm{d}x=\frac{1}{3}x^3+x^2+x+C.$$

■

Exercise 4.11 Find $\displaystyle\int \frac{3}{2\sqrt[3]{x^5}}\mathrm{d}x.$ ■

Polynomials divided by monomial terms work as well. A quick note before the example.

Note 4.5.6 Recall that for $a,m\in\mathbb{R}$,

$$a\cdot x^{-m}=\frac{a}{x^m}.$$

■ **Example 4.8** Find $\displaystyle\int \frac{x^4+x^3+2x^2+1}{x^2}\mathrm{d}x.$

As always, we must work smarter here, not harder. We can simply divide x^2 into each term in the numerator to find that

$$\int \frac{x^4+x^3+2x^2+1}{x^2}\mathrm{d}x=\int x^2+x+2+x^{-2}\mathrm{d}x.$$

Then we note that the integral of the sum is the sum of the integrals and

$$
\begin{aligned}
\int \frac{x^4+x^3+2x^2+1}{x^2}\mathrm{d}x &= \int x^2+x+2+x^{-2}\mathrm{d}x \\
&= \int x^2\mathrm{d}x+\int x^1\mathrm{d}x+\int 2x^0\mathrm{d}x+\int x^{-2}\mathrm{d}x \\
&= \tfrac{x^{2+1}}{2+1}+\tfrac{x^{1+1}}{1+1}+\tfrac{2x^{0+1}}{0+1}+\tfrac{x^{-2+1}}{-2+1}+C \\
&= \tfrac{x^3}{3}+\tfrac{x^2}{2}+\tfrac{2x^1}{1}+\tfrac{x^{-1}}{-1}+C \\
&= \tfrac{1}{3}x^3+\tfrac{1}{2}x^2+2x-\tfrac{1}{x}+C.
\end{aligned}
$$

Notice that we only added the constant of integration this time once at the end of the expression obtained after finding the antiderivative (or integral). This is conventionally how this is done.

■

Exercise 4.12 Find $\int x^4 - 3x^2 - \frac{3}{x^4} dx$. ■

This method works with roots as well, as long as we recall a fact about root notation (which is archaic and nonsensical and altogether terrible notation, but I digress).

Note 4.5.7 For $m \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\sqrt[n]{x^m} = x^{\frac{m}{n}}.$$

■ **Example 4.9** Find $\int \frac{2}{5\sqrt[4]{x^7}} dx$.

First, we note that $\int \frac{2}{5\sqrt[4]{x^7}} = \frac{2}{5x^{\frac{7}{4}}} = \frac{2}{5}x^{\frac{-7}{4}}$ by the note above. Then we have that

$$\begin{aligned}
\int \frac{2}{5\sqrt[4]{x^7}} dx &= \int \frac{2}{5}x^{\frac{-7}{4}} dx &&: \text{ Rewrite the root as an exponent.}\\
&= \frac{\frac{2}{5}x^{\frac{-7}{4}+1}}{\frac{-7}{4}+1} + C &&: \text{ Use the rules to take the antiderivative.}\\
&= \frac{\frac{2}{5}x^{\frac{-7}{4}+\frac{4}{4}}}{\frac{-7}{4}+\frac{4}{4}} + C &&: \text{ Common denominator for exponents.}\\
&= \frac{\frac{2}{5}x^{\frac{-3}{4}}}{\frac{-3}{4}} + C &&: \text{ Add exponents.}\\
&= \frac{-4}{3} \cdot \frac{2}{5}x^{\frac{-3}{4}} + C &&: \text{ Multiply by reciprocal to divide by a fraction.}\\
&= \frac{-8}{15}x^{\frac{-3}{4}} + C &&: \text{ Simplify fractions.}\\
&= \frac{-8}{15\sqrt[4]{x^3}} + C. &&: \text{ Rewrite the exponent as a root.}
\end{aligned}$$

■

Exercise 4.13 Find $\int \frac{3}{2\sqrt[3]{x^5}} dx$. ■

You may have noticed by now that the steps for these are pretty similar, but lists always help.

Note 4.5.8 — Evaluating Basic Antiderivatives. Following these steps will help you in learning how to take basic antiderivatives.

1. If there are roots, rewrite them as exponents.
2. If there are variables in denominators, write them using negative exponents instead.
3. Simplify everything you can before taking the antiderivative.
4. If the antiderivative is a sum or difference, take the antiderivative of each part separately.
5. Factor constants out of antiderivatives.
6. Simplify everything as you go.
7. Once you've taken the antiderivative, do not forget the constant of integration C in each step. Writing it once at the end of the expression is correct.

4.6 The Fundamental Theorem of Calculus

As we have seen in previous sections, using the definition to find an integral is cumbersome and tedious. While there are several methods to cover, there is a singular important theorem we would be remiss not to cover first. The Fundamental Theorem of Calculus (FTC) has two parts. The first part states that antiderivatives (or integrals) are the inverse of a derivative, no matter the function (provided it's continuous).

Definition 4.6.1 — The Fundamental Theorem of Calculus, Pt. I. If f is a continuous function on an interval $[a,b]$, and $F(x) = \int_a^x f(t)dt$, then over the interval $[a,b]$,

$$F'(x) = f(x).$$

The second part of the FTC details how to deal with integrals with definite bounds. While the actual definition of an integral in terms of the limit of a sum is spoken of later in the book, for now we state the definition.

Definition 4.6.2 — The Fundamental Theorem of Calculus, Pt. II. If $f(x)$ is continuous over $[a,b]$ and $F(x)$ is the antiderivative of $f(x)$, then

$$\int_b^a f(x)\mathrm{d}x = F(b) - F(a).$$

Before we begin, we need some additional notation.

Definition 4.6.3 — Bounding Shorthand. We write

$$[F(x)]_a^b = F(b) - F(a).$$

■ **Example 4.10** Find $\displaystyle\int_1^4 x^2 + 1\mathrm{d}x$.

First, find $F(x)$

$$F(x) = \int x^2 + 1\mathrm{d}x = \frac{1}{3}x^3 + x + C.$$

Then by the F.T.C. Pt. II,

$$\begin{aligned}
\int_a^b x^2 + 1\mathrm{d}x &= \left[\tfrac{1}{3}x^3 + x + C\right]_1^4 \\
&= \left(\tfrac{1}{3}(4)^3 + (4) + C\right) - \left(\tfrac{1}{3}(1)^3 + (1) + C\right) \\
&= \tfrac{64}{3} + 4 - \tfrac{1}{3} - 1 \\
&= \tfrac{64}{3} + \tfrac{12}{3} - \tfrac{1}{3} - \tfrac{3}{3} \\
&= \tfrac{72}{3} \\
&= 24.
\end{aligned}$$

Note 4.6.1 Notice how the constant of integration C vanished due to the subtraction. Since this will always happen with definite integrals, we usually omit the constant C if there are bounds.

■

■ **Example 4.11** Find $\displaystyle\int_{-3}^3 1 - 4x^3\mathrm{d}x$.

This process should be becoming super easy, barely an inconvenience. First, we find the indefinite integral (or antiderivative).

$$\begin{aligned}\int 1-4x^3\mathrm{d}x &= \int 1-\int 4x^3\mathrm{d}x \\ &= (x)-\left(\tfrac{4x^{3+1}}{3+1}\right)+C \\ &= x-x^4+C\end{aligned}$$

Then we note that

$$\int_{-3}^{3} 1-4x^3\mathrm{d}x = \left[x-x^4\right]_{-3}^{3}.$$

Therefore

$$\begin{aligned}\left[x-x^4\right]_{-3}^{3} &= [(3)-(3)^4]-[(-3)-(-3)^4] \\ &= [3-81]-[(-3)-81] \\ &= [-78]-[-84] \\ &= -78+84 \\ &= 6.\end{aligned}$$

■

■ **Example 4.12** Find $\int_{-2}^{2} x^3\mathrm{d}x$.

First, we find the the indefinite integral

$$\int x^3\mathrm{d}x = \frac{x^{3+1}}{3+1}+C = \frac{1}{4}x^4+C.$$

Then we evaluate to find

$$\begin{aligned}\int x^3\mathrm{d}x &= \left[\tfrac{1}{4}x^4\right]_{-2}^{2} \\ &= \left[\tfrac{1}{4}(2)^4\right]-\left[\tfrac{1}{4}\cdot(-2)^4\right] \\ &= \left[\tfrac{1}{4}\cdot 16\right]-\left[\tfrac{1}{4}\cdot 16\right] \\ &= 4-4 \\ &= 0.\end{aligned}$$

■

4.7 Area Between Two Curves

So far, we have only found the area between the x-axis and the curve generated by the function $f(x)$. It's easy enough to modify this method to find the area between two curves $f(x)$ and $g(x)$, where $f(x)$ is the topmost function.

Definition 4.7.1 — Area Between Two Curves. The area between the curves $f(x)$ and $g(x)$ on the interval $[a,b]$ is given by

$$A = \int_{a}^{b} f(x)-g(x)\mathrm{d}x,$$

provided $f(x) \geq g(x)$. That is, $f(x)$ is the greater function (or the upper function on the graph).

■ **Example 4.13** Find the area between the two functions $f(x) = x^2$ and $g(x) = 4$.

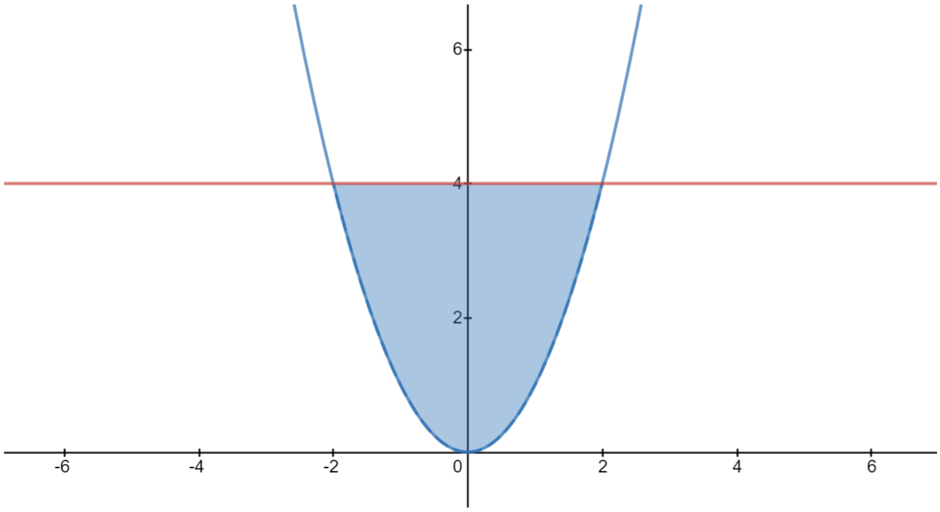

Figure 4.3: The two curves, illustrated.

So, note $g(x)$ is the top-most function, and $f(x)$ is the bottom-most function. Note also our bounds are $[-2,2]$. Then the area between f and g is simply

$$\begin{aligned}
\int_{-2}^{2} g(x) - f(x)\mathrm{d}x &= \int_{-2}^{2} 4 - x^2 \mathrm{d}x \\
&= \left[4x - \tfrac{1}{3}x^3\right]_{-2}^{2} \\
&= [4(2) - \tfrac{1}{3}(2)^3] - [4(-2) - \tfrac{1}{3}(-2)^3] \\
&= 8 - \tfrac{8}{3} - (-8 + \tfrac{8}{3}) \\
&= 8 - \tfrac{8}{3} + 8 - \tfrac{8}{3} \\
&= 16 - \tfrac{16}{3} \\
&= \tfrac{32}{3}.
\end{aligned}$$

■

Sometimes the bounds aren't given. In this case, we must find the intersection points of the two functions by setting them equal and solving.

■ **Example 4.14** Find the area between $f(x) = x^2$ and $g(x) = x^3$.

First, we find find where $x^3 = x^2$. So by subtracting x^2 from each side, we obtain $x^3 - x^2 = 0$. Then we know that $x^2(x-1) = 0$, and by the Zero Factor Property, we have that $x^2 = 0$ or $x - 1 = 0$. Then $x = 0$ or $x = 1$, and these are the bounds of our integral.

Now we must find which of the two functions is greater on the interval from $(0,1)$. Note that $x^2 > x^3$ on this entire interval. Then we proceed, letting $f(x) = x^2$ and $g(x) = x^3$.

$$\begin{aligned}
\int_{0}^{1} x^2 - x^3 \mathrm{d}x &= \left[\tfrac{1}{3}x^3 - \tfrac{1}{4}x^4\right]_0^1 \\
&= [\tfrac{1}{3}(1)^3 - \tfrac{1}{4}(1)^4] - [\tfrac{1}{3}(0)^3 - \tfrac{1}{4}(0)^4] \\
&= \tfrac{1}{3} - \tfrac{1}{4} \\
&= \tfrac{4}{12} - \tfrac{3}{12} \\
&= \tfrac{1}{12}.
\end{aligned}$$

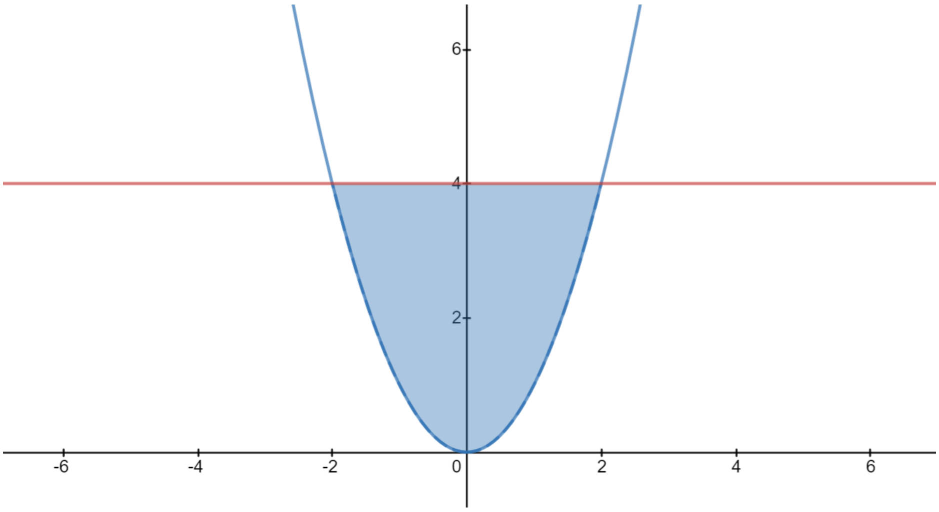

Figure 4.4: Figure caption

■

Exercise 4.14 Why is $x^2 > x^3$ on the interval $(0,1)$? ■

4.8 Common Integrals and Integral Properties

We have thusfar only given a formula for integrating monomials. There are several properties and common integrals that will be invaluable to us as we continue our study of Calculus.

4.8.1 Integral Properties

We will now state several properties of integrals, along with some common integrals that we should keep in mind. We omit the proofs, though they are not difficult:

Theorem 4.8.1 — Properties of Integration. Let $a,b,c,k \in \mathbb{R}$ with $a \leq c \leq b$ f and g continuous functions on the interval $I = [a,b]$. Further, let $o(x)$ be an odd function on I.

1. $\displaystyle\int_a^b 1\mathrm{d}x = b - a,$
2. $\displaystyle\int kf(x)\mathrm{d}x = k\int f(x)\mathrm{d}x,$
3. $\displaystyle\int (f(x) \pm g(x)) = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x,$
4. $\displaystyle\int_a^a f(x)\mathrm{d}x = 0,$
5. $\displaystyle\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x,$
6. $\displaystyle\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x,$
7. $\displaystyle\int_{-a}^a o(x) = 0.$

For a quick explanation of the properties above,

1. The definite integral over an interval of a constant function of height one is just the length of the interval.

2. The integral of a constant times a function is the constant times the integral of that function. That is, we can factor out constants from integrals.
3. The integral of a sum or difference is the sum or difference of the integrals. That is, we can 'break apart' integrals over addition or subtraction.
4. The area under the curve on an interval with length zero of a function is itself zero. That is, if the length of I is zero, the function has no area there.
5. Switching the bounds of integration changes the sign on the integral, and nothing else.
6. For some value c between a and b, we can 'break up' the interval into two (or more) parts at c.
7. The integral of an odd function over a symmetric interval is always zero (since there will always be as much positive area as there is negative).

4.8.2 Common Integrals

Recall that the first part of the F.T.C. tells us that the derivative of an integral is its integrand. Ridding ourselves of some of the cumbersome notation (and a bit of the nuance, which we will live with,) we have that

$$\int \frac{\mathrm{d}}{\mathrm{d}x} f(x)\mathrm{d}x = f(x) + C.$$

Using this logic, we'll recall that the derivative of $\sin x$ is $\cos x$. It stands to reason, by the fact mentioned above, that the integral of the derivative of $\sin x$ (which is $\cos x$) would therefore be $\sin x$. Therefore we have that

$$\int \cos x \mathrm{d}x = \sin x + C.$$

We use the same sort of logic to determine some other trigonometric derivatives. We'll list the derivatives here for convenience. Try and look for the pattern between the integrals and derivatives listed - it should be pretty evident.

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\csc x$	$-\csc x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$

To find the table for integrals, we simply switch the columns and move any negative signs over to the right-hand side.

Function	Integral
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\csc x \cot x$	$-\csc x$
$\sec x \tan x$	$\sec x$
$\csc^2 x$	$-\cot x$

R Notice how we do not yet have a formula for the integrals of $\sec x$, $\csc x$, $\tan x$, or $\cot x$. We will find those later!

By the same logic, since we know that the derivative of $\ln|x|$ is $1/x$, we know that

$$\int \frac{1}{x}\mathrm{d}x = \ln|x| + C.$$

The same logic gives us that

$$\int e^x \mathrm{d}x = e^x + C.$$

Therefore, combining what we know about antiderivatives, we have the following list of common integrals:

1. $\int \mathrm{d}x = x + C,$
2. $\int x^n \mathrm{d}x = \frac{x^{n+1}}{n+1} + C = \frac{1}{n+1}x^{n+1} + C,$
3. $\int \sin x \mathrm{d}x = \cos x + C,$
4. $\int \cos x \mathrm{d}x = -\sin x + C,$
5. $\int \sec^2 x \mathrm{d}x = \tan x + C,$
6. $\int \csc x \cot x \mathrm{d}x = \csc x + C,$
7. $\int \sec x \tan x \mathrm{d}x = \sec x + C,$
8. $\int \csc^2 x \mathrm{d}x = -\cot x + C,$
9. $\int \frac{1}{x}\mathrm{d}x = \ln|x| + C,$
10. $\int e^x \mathrm{d}x = e^x + C.$

Figure 4.5: A list of common integrals.

4.9 The Average Value of a Function

The average of a set of numbers is trivial to find. Simply add up the numbers and divide by the number of numbers you added. That is for a set $S = \{x_1, x_2, \ldots, x_n\}$ containing n elements such that each $x_i \in \mathbb{R}$, the average is simply

$$\bar{S} = \frac{x_1 + x_2 + \cdots + x_n}{n}.$$

Notice how we used $\overline{S}$, read 'S bar,' to denote the average of S. We will do the same for the average value of the function f, which we will denote $\overline{f}$. The derivation follows a road paved only with basic algebra, so we will proceed to simply stating the formula and following through with an example for clarity.

Definition 4.9.1 — The Average Value of a Function. Given a continuous function f defined on the closed interval $[a,b]$, the average value of the function $\overline{f}$ on $[a,b]$ is given by

$$\overline{f}|_a^b = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

■ **Example 4.15** Find the average value of the function $f(\theta) = \sin x$ on $[0,\pi]$.

For this, we note that $a = 0$, $b = \pi$, and our formula is thus

$$\overline{f}|_0^\pi = \frac{1}{\pi - 0}\int_0^\pi \sin x \mathrm{d}x = \frac{1}{\pi}[\cos x]_0^\pi = \frac{1}{\pi}[\cos(\pi) - \cos(0)] = \frac{1}{\pi}[-1-1] = -\frac{2}{\pi}$$

■

4.10 Integration by Substitution

In a standard Calculus class, there are five methods of integration. The author jokingly calls these the 'five deadly methods of integration,' if for no other reason than they are well-capable of slowly sapping one's sanity. The first of these is by far the most common and easiest, but first, we must remember the Chain Rule from Chapter 2. It said that given a composition of two functions f and g such that the range of g was in the domain of f, that the derivative of that composition was $f'(g(x))g'(x)$. That is, to take the derivative, simply take the derivative of the 'outer' function, plug the 'inner' function back in, then multiply by the derivative of this 'inner' function.

Since the first part of the F.T.C. states that the derivative of an integral is the integrand, we may use that fact, combined with the Chain Rule, to arrive at our first method. That is

$$\int f'(g(x))g'(x)\mathrm{d}x = f(g(x)) + C.$$

It is common for us to use the substitution $u = g(x)$, giving us the name u-substitution (or u-sub, for short). Doing that, we have

$$\int f'(u)u'\mathrm{d}x = f(u) + C.$$

Other than being mildly offensive if read out-of-context, this becomes a powerful tool for solving integrals in which there is a function composition with the derivative 'sitting' in the numerator as well (up to a constant). We will go through a few examples, since that will help to clarify things. Just keep in mind that we are trying to rewrite anything of the form $\int f'(g(x))g'(x)\mathrm{d}x$ as $\int f(u)\mathrm{d}u$.

■ **Example 4.16** Find $\int 2x\cos(x^2)\mathrm{d}x$.

For this and all integral problems henceforth, we are looking for patterns. In this particular problem, we notice that we have a function composition of x^2 with $\cos(x)$. Moreover (and more important,) we notice that the derivative of x^2, the function that is 'plugged in,' is sitting in the numerator outside of the composition! This is exactly what we want to use u-substitution. To do this process, we must choose our u. Our choice for u is not always straightforward, but it will almost always be the function that is the 'inner' function in the composition. In this case, we let $u = x^2$.

Now that we have our value for u (which remember is a function,) we must find our du. To do this, we simply take the derivative of u with respect to x. Therefore we have that

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x,$$

or if we multiply each side by dx, we have that

$$\mathrm{d}u = 2x\mathrm{d}x.$$

Then since we want to rewrite our integral as $\int u\mathrm{d}u$, we 'plug in' our u and du to obtain that

$$\int 2x\cos(x^2)\mathrm{d}x = \int \cos(u)\mathrm{d}u.$$

The $2x$ and dx terms became du, and the x^2 term became u. We know that $\int \cos(u)\mathrm{d}u = \sin u + C$ from the table from the previous problem, so we may be tempted to say that we're finished. However, our problem was given to us in terms of x, so we won't be finished until we 'un-substitute' our u back for x. Thankfully, we know that $u = x^2$, so our final answer is simply

$$\int 2x\cos(x^2)\mathrm{d}x = \sin(x^2) + C.$$

■

> **Note 4.10.1** It's worth noting that we can use the first part of the F.T.C. to check ourselves here. If we take the derivative of $\sin(x^2)+C$ using the Chain Rule, we will indeed end up with our integrand of $2x\cos(x^2)$. Neat!

We'll now do a couple more examples in full detail that are a bit more complicated, just so we can get the hang of things. In these, we have to manipulate a constant on the du term, but it's simple algebra.

■ **Example 4.17** Given the function $f(x) = \frac{3}{5}x\cos(x^2+1)$, find the integral with respect to x using the method of u-substitution.

Before we use u-substitution, we want to pull our $\frac{3}{5}$ out of the integral to make our math easier, giving us

$$\frac{3}{5}\int x\cos(x^2+1)\mathrm{d}x.$$

Now, u-substitution works with compositions of functions. We want to choose our u such that the derivative of u in some form is in the numerator of our integral. In our case, we want to choose x^2+1 as our u, take its derivative, and simplify. This gives us

$$\begin{aligned} u &= x^2+1 \\ \frac{\mathrm{d}u}{\mathrm{d}x} &= 2x \\ \mathrm{d}u &= 2x\mathrm{d}x. \end{aligned}$$

We don't have a $2x$ in our integral, but we do have an x. So in order to use u-substitution, we need to manipulate our u to give us x. We can do this by multiplying both sides by $\frac{1}{2}$, which makes our problem read

$$\frac{1}{2}du = xdx.$$

We now want to replace our x^2+1 with u and change our x and dx into the $\frac{1}{2}$du we just solved for. Now our problem looks like

$$\frac{3}{5}\int\frac{1}{2}\cos(u)\mathrm{d}u.$$

Next, we pull our fraction out of the integral and multiply it by $\frac{3}{5}$, giving us

$$\frac{3}{5}\left(\frac{1}{2}\right)\int\cos(u)\mathrm{d}u = \frac{3}{10}\int\cos(u)\mathrm{d}u.$$

It is important to note that we can use an elementary trig integral to finish solving this problem. Specifically, we will use

$$\int\cos(x)\mathrm{d}x = \sin(x)+C.$$

We are simply changing the names of the variables in the expression, so our problem reads

$$\frac{3}{10}\int\cos(u)\mathrm{d}u = \frac{3}{10}[\sin(u)]+C.$$

Lastly, we will plug our original function for u back in, giving this section a final answer of

$$\frac{3}{10}\sin(x^2+1)+C.$$

■

■ **Example 4.18** Given $f(t) = -\frac{\sqrt{2}\cos 2t}{\sqrt{\sin 2t+2}}$, find the integral using u-substitution.
First, we must rewrite our function as

$$\int -\frac{\sqrt{2}\cos 2t}{\sqrt{\sin 2t+2}}\mathrm{d}t.$$

For the sake of simplicity, we can pull out the constant of $-\sqrt{2}$. This results in

$$-\sqrt{2}\int \frac{\cos 2t}{\sqrt{\sin 2t+2}}\mathrm{d}t.$$

In order to progress, we must choose a u for u-substitution. By looking at the problem, we can determine that $u = \sin 2t + 2$ since its derivative is in the numerator. In order to be certain that this is in fact our u- value, we must take the derivative of our suspected u. Since there is a function composition, we begin by using the Chain Rule

$$\begin{aligned} u &= \sin 2t + 2 \\ \mathrm{d}u &= 2\cos(2t)\mathrm{d}t \\ \frac{1}{2}\mathrm{d}u &= \cos(2t)\mathrm{d}t. \end{aligned}$$

Since our du in the problem is missing the constant, we must do some simple algebra to find out what $\cos(2t)\mathrm{d}t$ is in terms of $\mathrm{d}u$. We must now substitute in our u value and our $\mathrm{d}u$ value. We also notice that $\frac{1}{2}$ is a constant, we can pull it out and multiply it by $-\sqrt{2}$.

$$-\sqrt{2}\int \frac{1}{2}\frac{1}{\sqrt{u}}\mathrm{d}u = -\frac{\sqrt{2}}{2}\int \frac{1}{\sqrt{u}}\mathrm{d}u.$$

Now we note that $1/\sqrt{u} = 1/u^{\frac{1}{2}} = u^{-\frac{1}{2}}$. Therefore

$$-\frac{\sqrt{2}}{2}\int \frac{1}{\sqrt{u}}\mathrm{d}u = -\frac{\sqrt{2}}{2}\int u^{-\frac{1}{2}}\mathrm{d}u = -\frac{\sqrt{2}}{2}\left[\frac{1}{-\frac{1}{2}+1}u^{-\frac{1}{2}+1}\right]+C = -\frac{\sqrt{2}}{2}\left[\frac{1}{\frac{1}{2}}u^{\frac{1}{2}}\right]+C.$$

Simplifying things down gives us our final answer of

$$-\sqrt{2}\sqrt{u}+C = -\sqrt{2}\sqrt{\sin 2t+2}+C = -\sqrt{2(\sin 2t+2)}+C = -\sqrt{2\sin 2t+4}+C.$$

Therefore

$$\int -\frac{\sqrt{2}\cos 2t}{\sqrt{\sin 2t+2}}\mathrm{d}t = -\sqrt{2\sin 2t+4}+C.$$

■

4.11 Integration by Parts

The second of the five standard integration methods is called Integration by Parts. It more or less feels like u-substitution on steroids, but has different techniques to handle it.

In much the same way that u-substitution is based on the Chain Rule (but in reverse, more or less), the method of Integration by Parts (IBP) is based on the Product Rule. We recall that now, except instead of our standard choice of functions f and g, we choose u and v, which helps avoid confusion down the road.

Theorem 4.11.1 — The Product Rule - Lagrange Notation. For differentiable functions u and v,

$$[uv]' = u'(x)](v(x)) + (u(x))[v'(x)],$$

or more condensed,

$$[uv]' = u'v + uv'.$$

To derive the formula for IBP, we rearrange the product rule as such

$$[uv]' = u'v + uv' \implies uv' = [uv]' - u'v.$$

Now we simply integrate on both sides

$$\int uv'\mathrm{d}x = \int [uv]'\mathrm{d}x - \int u'v\mathrm{d}x.$$

Note 4.11.1 It helps to recall that

$$u' = \frac{\mathrm{d}u}{\mathrm{d}x}$$

and that

$$v' = \frac{\mathrm{d}v}{\mathrm{d}x}.$$

So that

$$\int uv'\mathrm{d}x = \int [uv]'\mathrm{d}x - \int \frac{\mathrm{d}u}{\mathrm{d}x}v\mathrm{d}x$$

becomes

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = \int [uv]'\mathrm{d}x - \int \frac{\mathrm{d}u}{\mathrm{d}x}v\mathrm{d}x.$$

The dx terms cancel so that we have just

$$\int u\mathrm{d}v = \int [uv]'\mathrm{d}x - \int v\mathrm{d}u.$$

We note by the first part of the FTC, we have that

$$\int [uv]'\mathrm{d}x = uv,$$

so that our whole formula simplifies down into what we call Integration by Parts.

Definition 4.11.1 — Integration by Parts. For two continuously differentiable functions $u(x)$ and $v(x)$,

$$\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$$

4.11.1 Using Integration by Parts

The method for actually using the IBP formula is more of a practiced art than a hard and prescriptive rule. Generally, the first step to a problem involving IBP is making sure the problem is actually able to be solved by the technique. This usually involves seeing if the integrand is a product of two functions. One of the functions should vanish or repeat if its derivative is taken, and the other function should be one that is not an absolute nightmare to integrate.

R When we say 'vanish or repeat if its derivative is taken,' we mean something like $f(x) = x^3$ for the former, where eventually its derivative is zero; and something like $f(x) = \sin x$ for the latter, where $\sin x$ is a derivative on down the line (around four derivatives or so).

As always, this method only becomes clear after quite a bit of practice, so let us get on with that.

■ **Example 4.19** Find the following integral:

$$\int xe^x\mathrm{d}x.$$

Clearly the integrand xe^x is a product of two functions. We must choose which of these is our u and which of these is the functions is our dv, since we want something of the form

$$\int u\mathrm{d}v$$

so that IBP can apply. As mentioned in the remark, we want one of these functions (our u) to be a function that vanishes after repeated derivatives or repeats (or at least does something useful). In this case, it should be pretty obvious that between the two choices of $u = x$ or $u = e^x$, that $u = x$ is clearly the better choice. Of course, that leaves us with $\mathrm{d}v = e^x\mathrm{d}x$, which means we have to integrate it. It is at this point that we make sure our dv is something we can actually integrate. Thankfully, we can. We can now set up our integration by parts with our $u = x$ and $\mathrm{d}v = e^x\mathrm{d}x$. If you want to think of dv as a variable here, go ahead. It really is just a tiny nudge of v, anyway. Our next step is to take the derivative of u (since we have u and need du for the IBP formula), and to integrate dv (since we have dv and need v for the IBP formula).

$$\begin{aligned} \text{Let } u = x \quad & \& \quad \mathrm{d}v = e^x\mathrm{d}x \\ \text{Then } \mathrm{d}u = \mathrm{d}x \quad & \& \quad v = e^x. \end{aligned}$$

Then we simply take our formula and replace parts (hence the name). To wit:

$$\int xe^x\mathrm{d}x = \underbrace{\qquad}_{u}\underbrace{\qquad}_{v} - \int \underbrace{\qquad}_{v}\underbrace{\qquad}_{\mathrm{d}u} = uv - \int v\mathrm{d}u = \underbrace{x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}.$$

All that remains is to note that the integral of e^x is simply itself plus a constant of integration, ending us with

$$\int xe^x\mathrm{d}x = xe^x - e^x + C.$$

■

Note 4.11.2 A careful reader may have notice in our little breakout box when we took the integral of our dv that we did not obtain $v = e^x + C$. This is because when we input the v into our formula for IBP, we do so twice and they will always differ in sign (because of the minus operation in the IBP formula itself). Thus they would simply cancel. We only add on the customary constant at the end of the process for this reason.

■ **Example 4.20** Find the integral for

$$\int x^2 e^{2x}.$$

We proceed much the same way as previously discussed. We choose $u = x^2$ since it will eventually vanish, leaving d$v = e^{2x}$dx. Then

$$\begin{aligned} \text{Let } u = x^2 \quad &\& \ \mathrm{d}v = e^{2x}\mathrm{d}x \\ \text{Then } \mathrm{d}u = 2x\mathrm{d}x \ &\& \quad v = \tfrac{1}{2}e^{2x}. \end{aligned}$$

Then we have that

$$\int x^2 e^{2x}\mathrm{d}x = \underbrace{x^2}_{u}\underbrace{\frac{1}{2}e^{2x}}_{v} - \int \underbrace{\frac{1}{2}e^{2x}}_{v}\underbrace{2x\mathrm{d}x}_{\mathrm{d}u} = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\mathrm{d}x.$$

Now we note that we have another integral to solve within this problem, and it is also able to be solved by IBP! We will simply carry the terms preceding this integral down as we always have, then

$$\begin{aligned} \text{Let } u = x \quad &\& \ \mathrm{d}v = e^{2x}\mathrm{d}x \\ \text{Then } \mathrm{d}u = \mathrm{d}x \ &\& \quad v = \tfrac{1}{2}e^{2x}. \end{aligned}$$

Thus

$$\int x^2e^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \left[\underbrace{x}_{u}\underbrace{\frac{1}{2}e^{2x}}_{v} - \int \underbrace{\frac{1}{2}e^{2x}}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}\right].$$

This simplifies to

$$\int x^2e^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\int e^{2x}\mathrm{d}x.$$

All that is left is to take the integral of e^{2x}, which we have already done twice, and we arrive at our final answer of

$$\int x^2e^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \int xe^{2x}\mathrm{d}x = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{2}\left[\frac{1}{2}e^{2x}\right] = \frac{1}{2}x^2e^{2x} - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C.$$

■

In general, there exists a recursive formula for integrating integrands of the form $x^n e^{rx}$. We will use this in the next section, but we present it while it is still fresh.

Definition 4.11.2 Given $r \in \mathbb{R}$ and $n \in \mathbb{N}$, the following holds:

$$\int x^n e^{rx} \mathrm{d}x = \frac{1}{rx^n} e^{rx} - \frac{n}{r} \int x^{n-1} e^{rx} \mathrm{d}x \, .$$

Example 4.21 Find

$$\int x^2 \cos x \mathrm{d}x.$$

As always, we have to assign our u and $\mathrm{d}v$. If we choose $u = \cos x$, that leaves $\mathrm{d}v = x^2$. When we integrate $\mathrm{d}v$, we'd get that v is now a third-degree term, which moves us farther from a solution. That means that we must proceed as follows:

$$\begin{aligned} \text{Let } u &= x^2 \quad \& \;\; \mathrm{d}v = \cos x \mathrm{d}x \\ \text{Then } \mathrm{d}u &= 2x\mathrm{d}x \;\; \& \quad v = \sin x. \end{aligned}$$

Then it is just a fill-in-the-blanks exercise as usual

$$\int x^2 \cos x \mathrm{d}x = \underbrace{\quad}_{u}\underbrace{\quad}_{v} - \int \underbrace{\quad}_{v}\underbrace{\quad}_{\mathrm{d}u} = \underbrace{x^2}_{u}\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\underbrace{2x\mathrm{d}x}_{\mathrm{d}u} = x^2 \sin x - 2\int x \sin x \mathrm{d}x.$$

It is generally a good idea to factor out constants as often as possible, and try to keep terms in the same order. In this case, the order is a monomial followed by the trigonometric function. This bit of bookkeeping helps keep the work manageable, especially while you're learning. We note now that $x \sin x$ is another integrand that is accomplished via IBP, so we once again invoke it.

$$\begin{aligned} \text{Let } u &= x \quad \& \;\; \mathrm{d}v = \sin x \mathrm{d}x \\ \text{Then } \mathrm{d}u &= \mathrm{d}x \;\; \& \quad v = -\cos x. \end{aligned}$$

Then we have that

$$\int x^2 \cos x \mathrm{d}x = \underbrace{x^2}_{u}\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\underbrace{2x\mathrm{d}x}_{\mathrm{d}u} = x^2 \sin x - 2\left[\underbrace{\quad}_{u}\underbrace{\quad}_{v} - \int \underbrace{\quad}_{v}\underbrace{\quad}_{\mathrm{d}u}\right].$$

Filling in our latest blanks we end up with

$$\int x^2 \cos x \mathrm{d}x = \underbrace{x^2}_{u}\underbrace{\sin x}_{v} - \int \underbrace{\sin x}_{v}\underbrace{2x\mathrm{d}x}_{\mathrm{d}u} = x^2 \sin x - 2\left[\underbrace{x}_{u}\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\underbrace{\mathrm{d}x}_{\mathrm{d}u}\right].$$

Notice all of the negative and minus signs in this problem. Be very careful with these! We go ahead and clean those up to obtain that

$$\int x^2 \cos x \mathrm{d}x = x^2 \sin x + 2\left[x \cos x - \int \cos x \mathrm{d}x\right].$$

There is nothing left to do but note that the integral of $\cos x$ is simply $\sin x$, so we distribute the constant term and end with

$$\int x^2 \cos x \mathrm{d}x = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

■

■ **Example 4.22** Evaluate the integral

$$\int e^x \cos x \mathrm{d}x.$$

Here is an example where no matter your choice of u, it will never vanish. However, a careful reader would note that there was another criteria for u-choice, which was that the derivative repeats eventually. In this case, both e^x and $\cos x$ repeat eventually, with the former repeating itself after just one derivative, and the latter taking four to cycle back around. For this reason, it does not matter which you assign to u. Without loss of generality,

$$\begin{aligned} \text{Let } u &= \cos x \quad \& \ \mathrm{d}v = e^x \mathrm{d}x \\ \text{Then } \mathrm{d}u &= -\sin x \mathrm{d}x \ \& \quad v = e^x. \end{aligned}$$

so that

$$\int e^x \cos x \mathrm{d}x = \underbrace{\qquad}_{u}\underbrace{\qquad}_{v} - \int \underbrace{\qquad}_{v}\underbrace{\qquad}_{\mathrm{d}u} = \underbrace{\cos x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{(-\sin x)\mathrm{d}x}_{\mathrm{d}u}.$$

Meaning that we have that

$$\int e^x \cos x \mathrm{d}x = e^x \cos x + \int e^x \sin x \mathrm{d}x.$$

We must again use IBP. Ensure this time you make the same type of function your u, so that you do not reverse the work you just did. We choose a trigonometric function the first time, so we will do so again this time.

$$\begin{aligned} \text{Let } u &= \sin x \quad \& \ \mathrm{d}v = e^x \mathrm{d}x \\ \text{Then } \mathrm{d}u &= \cos x \mathrm{d}x \ \& \quad v = e^x. \end{aligned}$$

Then we now have that

$$\int e^x \cos x \mathrm{d}x = e^x \cos x + \int e^x \sin x \mathrm{d}x = e^x \cos x + \left[\underbrace{\qquad}_{u}\underbrace{\qquad}_{v} - \int \underbrace{\qquad}_{v}\underbrace{\qquad}_{\mathrm{d}u}\right].$$

We note that the brackets do exactly nothing here, so we drop those and fill out our IBP to obtain

$$\int e^x \cos x \mathrm{d}x = e^x \cos x + \underbrace{\sin x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{\cos x \mathrm{d}x}_{\mathrm{d}u} = e^x \cos x + e^x \sin x - \int e^x \cos x \mathrm{d}x.$$

We now note that the integral we have on the right-hand side of the equation is identical to the one of the left-hand side! This is even more noticeable if we eliminate the intermediate step to see that

$$\int e^x \cos x \mathrm{d}x = e^x \cos x + e^x \sin x - \int e^x \cos x \mathrm{d}x.$$

We therefore add the integral on both sides to obtain

$$2 \int e^x \cos x \mathrm{d}x = e^x \cos x + e^x \sin x.$$

Meaning we simply divide both sides by the constant to arrive at our solution of

$$\int e^x \cos x \mathrm{d}x = \frac{e^x \cos x + e^x \sin x}{2} + C.$$

■

4.12 Bryant's Formula

There is a very simple recursive formula for $\int e^{rx}x^n \mathrm{d}x$ that is derived by the method of Integration by Parts:

$$\int x^n e^{rx} \mathrm{d}x = \frac{1}{r}x^n e^{rx} - \frac{n}{r}\int x^{n-1}e^{rx}\mathrm{d}x. \tag{4.1}$$

As all things in mathematics, we wish to see if there is a pattern within.

4.12.1 Derivation of a Summation Formula

Evaluating the integral on the right-hand side of the formula above expands it into the following:

$$\int e^{rx}x^n \mathrm{d}x = \frac{1}{r}e^{rx}x^n - \frac{n}{r}\left(\frac{1}{r}x^{n-1}e^{rx} - \frac{(n-1)}{r}\int x^{n-2}e^{rx}\mathrm{d}x\right).$$

Distributing the n term through the large parenthesis and evaluating the integral, we see a falling factorial term $n(n-1)(n-2)\cdots(n-k+1)$ after k derivations develop. So we have that $\int e^{rx}x^n \mathrm{d}x$ can be expressed as

$$e^{rx}x^n - \frac{n}{r^2}x^{n-1}e^{rx} - \frac{n(n-1)}{r^2}\left(\frac{1}{r}x^{n-2}e^{rx} - \frac{(n-2)}{r}\int x^{n-3}e^{rx}\mathrm{d}x\right).$$

Continuing the pattern, it becomes obvious that each term, including the last, will have an e^{rx} multiplied by it.

> **Theorem 4.12.1 — Bryant's Theorem.** Noting that the falling factorial notation $n^{\underline{i}}$, read "n to the i falling," can be expressed as $n(n-1)\cdots(n-i+1) = \frac{n!}{(n-i)!}$, and restricting $n \in \mathbb{N}$ and $r \in \mathbb{R}$, with $r \neq 0$ we can define the summation formula as such:
>
> $$\int x^n e^{rx}\mathrm{d}x = \frac{e^{rx}}{r^{n+1}}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right) + C.$$

To simplify the work, we define a new function $\sigma_r^n(x)$:

$$\sigma_r^n(x) := \left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right).$$

This makes the summation formula for $\int e^{rx}x^n \mathrm{d}x$ into something a bit less cumbersome to read:

$$\int e^{rx}x^n \mathrm{d}x = \frac{e^{rx}}{r^{n+1}}\sigma_r^n(x) + C.$$

From formula 4.1 above, consider the trivial case when $n = 0$. It should be easy to see that if $n = 0$, the integrand reduces to e^{rx}, whose derivative is simple to find:

$$\int e^{rx}\mathrm{d}x = \frac{1}{r}e^{rx} + C.$$

Now let us see what happens when we apply our conjectured formula using the special value of $n = 0$. As with most applications of $n!$, we will let $0! = 1$. So for $n = 0$

$$\int x^0 e^{rx} \mathrm{d}x = \frac{e^{rx}}{r^{0+1}} \sigma_r^0(x) + C = \frac{e^{rx}}{r} \left(\sum_{i=0}^{0} (-1)^i 0^{\underline{i}}\, r^{0-i} x^{0-i} \right) + C.$$

Evaluating this leaves

$$\frac{e^{rx}}{r}((-1)^0 0^{\underline{0}}\, r^0 x^{0-0}) + C = \frac{e^{rx}}{r}((1)(1)(1)(1)) + C = \frac{1}{r} e^{rx} + C.$$

For $n = 1$ through $n = 5$, we have

Integral	Evaluation
$\int e^{rx} x^1 \mathrm{d}x$	$\frac{e^{rx}}{r^2}(rx - 1) + C$
$\int e^{rx} x^2 \mathrm{d}x$	$\frac{e^{rx}}{r^3}(r^2x^2 - 2rx + 2) + C$
$\int e^{rx} x^3 \mathrm{d}x$	$\frac{e^{rx}}{r^4}(r^3x^3 - 3r^2x^2 + 6rx - 6) + C$
$\int e^{rx} x^4 \mathrm{d}x$	$\frac{e^{rx}}{r^5}(r^4x^4 - 4r^3x^3 + 12r^2x^2 - 24rx + 24) + C$
$\int e^{rx} x^5 \mathrm{d}x$	$\frac{e^{rx}}{r^6}(r^5x^5 - 5r^4x^4 + 20r^3x^3 - 60r^2x^2 + 120rx - 120) + C$

Figure 4.6: Evaluations from $n = 1$ to $n = 5$.

Compare the constants with the following values of $n^{\underline{i}} = \frac{n!}{(n-i)!}$, remembering that $0 \le i \le n$:

i	0	1	2	3	4	5
$n = 1$	1	1				
$n = 2$	1	2	2			
$n = 3$	1	3	6	6		
$n = 4$	1	4	12	24	24	
$n = 5$	1	5	20	60	120	120

Figure 4.7: Values of falling factorials.

Note that these values correspond exactly to the constants in front of each of x^{n-i}. Together with the knowledge that the signs alternate, we can see how each term could be represented by $(-1)^i n^{\underline{i}}\, r^{n-i} x^{n-i}$ if we let our value of i run from 0 to n.

4.12.2 A Proof of the Formula

First, we reiterate the first part of the Fundamental Theorem of Calculus.

Theorem 4.12.2 — First Fundamental Theorem of Calculus. If f a continuous function on an open interval I and a any point in I, if F is defined by

$$F(x) = \int_a^x f(t)\mathrm{d}t,$$

then $F'(x) = f(x)$ at each point in I.

Let $F(x) = \frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)$ and $f(x) = x^n e^{rx}$. If we can show $F'(x) = f(x)$, then by the above theorem we will have proven the conjecture.

Note 4.12.1 It is important that the reader realize that we are not proving the First Fundamental Theorem of Calculus. We are simply using the FTC to prove our conjecture.

Proof. To begin, we will define the function $\sigma_r^n(x)$ as such

$$\sigma_r^n(x) := \left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i} x^{n-i}\right).$$

Now we will derive $[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)]'$ by the Product Rule as follows:

$$\left[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)\right]' = \left[\frac{re^{rx}}{r^{n+1}}\sigma_r^n(x) + \frac{e^{rx}}{r^{n+1}}(\sigma_r^n(x))'\right] = e^{rx}\left[\frac{1}{r^n}\sigma_r^n(x) + \frac{1}{r^{n+1}}(\sigma_r^n(x))'\right].$$

Note that our strategy is to prove that $[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)]' = e^{rx}x^n$. Since we just showed that there exists an e^{rx} term on each side, we can cancel each by division and proceed, noting that we are trying to show that $\sigma_r^n(x) + (\sigma_r^n(x))' = x^n$. It therefore it follows that we should find $(\sigma_r^n(x))'$:

$$(\sigma_r^n(x))' = \frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i} x^{n-i}\right).$$

Remembering that the derivative of a finite sum is the sum of the derivatives we have

$$(\sigma_r^n(x))' = \left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}\frac{\mathrm{d}}{\mathrm{d}x}x^{n-i}\right).$$

So by the Power Rule this evaluates to

$$(\sigma_r^n(x))' = \left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, r^{n-i}(n-i)x^{n-i-1}\right).$$

The careful reader may have noticed that the nth term was effectively canceled when the derivative was taken. This is because it was the topmost term and for $i = n$ was a constant. The derivative of a constant is zero. With that said, putting this all back together, we have that $[e^{rx}\sigma_r^n(x)]'$ can be expressed as

$$\left[\frac{1}{r^n}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i} x^{n-i}\right) + \frac{1}{r^{n+1}}\left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, r^{n-i}(n-1)x^{n-i-1}\right)\right].$$

We may now recall that what we are dealing with is assumed to be x^n. Setting our expression equal to x^n sees us arrive at what is the key to our proof, which is that

$$\frac{1}{r^n}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i} x^{n-i}\right) + \frac{1}{r^{n+1}}\left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\, r^{n-i}(n-1)x^{n-i-1}\right) = x^n. \tag{4.2}$$

However, we cannot use this result just yet, since we have yet to prove that our formula is actually true. So compacting this down, we must now prove that

$$\frac{1}{r^n}\sigma_r^n(x) + \frac{1}{r^{n+1}}(\sigma_r^n(x))' = x^n.$$

To simplify our work, we will use these identities to distribute each of the fractions through the sums in equation 4.2 to see that we are now trying to prove that

$$\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\,\frac{1}{r^i}x^{n-i}\right)+\left(\sum_{i=0}^{n-1}(-1)^i n^{\underline{i}}\,\frac{1}{r^{i+1}}(n-1)x^{n-i-1}\right)=x^n.$$

To do this, we will first note that, for $k=n-i$,

$$\frac{r^{n-i}}{r^n}=r^{-i}=\frac{1}{r^i}=\frac{1}{r^{n-k}}.$$

Let $k=n-i$. Note that as i ranges from 0 to n, k ranges from n to 0. Since we are working with a summation, k also ranges from 0 to n. So rewriting $\sigma_r^n(x)$ with our new variable k, noting again that

$$n^{\underline{i}}=n(n-1)\ldots(n-i+1)=\frac{n!}{(n-i)!}.$$

From this, we obtain

$$\frac{1}{r^n}\sigma_r^n(x)=\frac{1}{r^n}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}r^kx^k=\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\frac{1}{r^{n-k}}x^k.$$

Recall that we found $(\sigma_r^n(x))'$ earlier. Note that when we change our variables using $k=n-i$, we reverse the order of the polynomial. Because of this, instead of running from $i=0$ to $n-1$ in order to omit the last term $i=n$, we run from $k=1$ to n to omit the first term $k=0$. Knowing this, we now have

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))'=\frac{1}{r^{n+1}}\sum_{k=1}^{n}(-1)^{n-k}\frac{n!}{k!}r^kkx^{k-1}.$$

Canceling the k with the first term in $k!$ yields

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))'=\frac{1}{r^{n+1}}\sum_{k=1}^{n}(-1)^{n-k}\frac{n!}{(k-1)!}r^kx^{k-1}.$$

To make this easier to work with, we will shift the index down one to get

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))'=\frac{1}{r^{n+1}}\sum_{k=0}^{n-1}(-1)^{n-(k+1)}\frac{n!}{k!}r^{k+1}x^k.$$

Factoring out a negative term and reincorporating the n^{th} terms gives

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))'=\frac{1}{r^{n+1}}\left[\left(-\sum_{k=0}^{n-1}(-1)^{n-k}\frac{n!}{k!}r^{k+1}x^k\right)-\frac{n!}{n!}r^{n+1}x^n+\frac{n!}{n!}r^{n+1}x^n\right].$$

We will now note the identity

$$\frac{r^{k+1}}{r^{n+1}}=r^{-(n-k+1)-1}=r^{-(n-k-1+1)}=\frac{1}{r^{n-k}}.$$

Distributing through the outermost fraction allows us to cancel both of the r^{n+1} terms attached to their respective x^n terms and gives that

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))'=\left[\left(-\sum_{k=0}^{n-1}(-1)^{n-k}\frac{n!}{k!}\frac{1}{r^{n-k}}x^k\right)-\frac{n!}{n!}x^n+\frac{n!}{n!}x^n\right].$$

Collecting the terms gives us that

$$\frac{1}{r^{n+1}}(\sigma_r^n(x))' = \left(-\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!}\frac{1}{r^{n-k}}x^k\right) + x^n.$$

Noting that we have just shown that $\frac{1}{r^{n+1}}(\sigma_r^n(x))' = -\frac{1}{r^n}\sigma_r^n(x) + x^n$, surely

$$\frac{1}{r^n}\sigma_r^n(x) + \frac{1}{r^{n+1}}(\sigma_r^n(x))' = x^n.$$

Therefore $[\frac{e^{rx}}{r^{n+1}}\sigma_r^n(x)]' = x^n e^{rx}$ and by the Second Fundamental Theorem of Calculus,

$$\int x^n e^{rx}\mathrm{d}x = \frac{e^{rx}}{r^{n+1}}\left(\sum_{i=0}^{n}(-1)^i n^{\underline{i}}\, r^{n-i}x^{n-i}\right) + C. \tag{4.3}$$

■

4.12.3 The Definite Integral Using Bryant's Formula

Now we call on the other part of the FTC:

Theorem 4.12.3 — Second Fundamental Theorem of Calculus. If f is continuous on the closed interval $[a,b]$ and F is the indefinite integral of f on $[a,b]$ then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

We have found a closed-form expression which is equal to the indefinite integral of $x^n e^{rx}$. Then it would make sense that

$$\int_a^b e^{rx}x^n\mathrm{d}x = \frac{1}{r^{n+1}}[e^{br}\sigma_r^n(b) - e^{ar}\sigma_r^n(a)]. \tag{4.4}$$

Appendices

Appendix I: Useful Formulas

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}$$

$$\tan\frac{\theta}{2} = \frac{1-\cos\theta}{\sin\theta}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

In the polynomial $ax^2+bx+c=0$, with $a \neq 0$,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Appendix II: Course Alignment

This text is aligned with the Virginia Community College System's (VCCS) standards for MTH 263. As the VCCS does not number their course objectives, they are numbered below. They have also been lightly edited to conform to the style of this text and rules of basic English. The full and current text can be found at `https://courses.vccs.edu/courses/MTH263-CalculusI/detail`.

1. Limits
   (a) Differentiate between the limit and the value of a function at a point.
   MET IN SECTION(S): 1.2.2
   (b) Find the limit of a function by numerical, graphical, and analytic methods.
   MET IN SECTION(S): 1.2.3, 1.2.4, 1.2.5
   (c) Apply limit laws.
   MET IN SECTION(S): 1.4
   (d) Calculate one-sided limit of a function.
   MET IN SECTION(S): 1.2.2, 1.2.4
   (e) Prove the existence of a limit using precise definition of the limit.
   MET IN SECTION(S): 1.3
   (f) Determine the continuity of a function.
   MET IN SECTION(S): 1.6
   (g) Calculate vertical and horizontal asymptotes using limits.
   MET IN SECTION(S): 1.2.5, 1.5
2. Derivatives and Differentiation Rules
   (a) Define derivatives and rates of change.
   MET IN SECTION(S): 2.1, 2.1.1
   (b) Compute derivatives of basic functions using the definition of the derivative.
   MET IN SECTION(S): 2.1.1
   (c) Differentiate polynomial, rational, radical, exponential, and logarithmic functions.
   MET IN SECTION(S): 2.3, 2.8
   (d) Find the equation of a tangent line using the derivative.
   MET IN SECTION(S): 2.1.2
   (e) Differentiate trigonometric functions.
   MET IN SECTION(S): 2.5
   (f) Apply the product, quotient, chain rules.
   MET IN SECTION(S): 2.6, 2.7, 2.9
   (g) Apply implicit differentiation and find derivatives of inverse trigonometric functions.
   MET IN SECTION(S): 2.12, 2.10
   (h) Apply the concept of rates of change to the natural and social sciences.
   MET IN SECTION(S): 3.1, 3.2
   (i) Apply the concept of related rates.
   MET IN SECTION(S): 3.3
   (j) Define hyperbolic functions and their derivatives.
   MET IN SECTION(S): 2.11
   (k) Find the linear approximation of a function at a given point.
   MET IN SECTION(S): 2.1.3
3. Applications of Differentiation
   (a) Calculate the local and absolute maximum and minimum values of a function.
   MET IN SECTION(S): 3.4
   (b) Apply Rolle's Theorem and the Mean Value Theorem to study properties of a function.
   MET IN SECTION(S): 3.7

(c) Find critical points and intervals of increasing and decreasing values of a function.
MET IN SECTION(S): 3.6
(d) Find points of inflection and intervals of different concavities.
MET IN SECTION(S): 3.6
(e) Sketch a curve for a given function.
MET IN SECTION(S): 3.10
(f) Apply rules of differentiation to solve optimization problems.
MET IN SECTION(S): 3.11
(g) Find antiderivatives for basic functions using knowledge of derivatives.
MET IN SECTION(S): 4.5

4. Integrals
   (a) Relate areas to definite integrals using sigma notation, Riemann Sums, and limits.
   MET IN SECTION(S): 4.1, 4.2, 4.3, 4.4
   (b) Apply the Fundamental Theorem of Calculus to find definite integrals and derivatives.
   MET IN SECTION(S): 4.6
   (c) Find indefinite integrals of polynomials and basic trig and exponential functions.
   MET IN SECTION(S): 4.5, 4.8.2
   (d) Apply the Net Change Theorem.
   MET IN SECTION(S): 4.6
   (e) Perform integration using substitution.
   MET IN SECTION(S): 4.10
   (f) Find areas between curves.
   MET IN SECTION(S): 4.7
   (g) Find the average value of a function.
   MET IN SECTION(S): 4.9

Bibliography

Books

Articles

Index of Key Terms